[24] J. LERAY and J. SCHAUDER, *Topologie et equations fonctionelles*, Ann. Sci. Ec. Norm. Sup., 51 (1934), 45-78.

[25] M. NAGUMO, *A theory of degree of mapping based on infinitesimal analysis*, Amer. Jour. Math., 73 (1951), 485-496.

[26] — —, *Degree of mapping in convex linear topological spaces*, Amer. Jour. Math., 73 (1951), 497-511.

[27] M. Z. NASHED and J. S. WONG, *Some variants of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations*, Jour. of Math. and Mech., 18 (1969), 767-777.

[28] R. D. NUSSBAUM, *The fixed point index and asymptotic fixed point theorems for k-set-contractions*, Bull. Amer. Math. Soc, 75 (1969), 490-495.

[29] — —, *The radius of the essential spectrum*, Duke Math. Jour., 38 (1970), pp. 473-478.

[30] — —, *A generalization of the Ascoli theorem and an application to functional differential equations*, Jour. Math. Analysis and its Appl., 35 (1971), pp. 600-610.

[31] — —, *The fixed point index and fixed point theorems for k-set-contractions*, Unpublished Ph D. dissertation (University of Chicago, 1969).

[32] — —, *Asymptotic fixed point theorems for local condensing mops*, Math. Annalen, 191 (1971), pp. 181-195.

[33] — —, *Degree theory for local condensing maps*, Jour. Math. Analysis and its Appl. (to appear).

[34] R. S. PALAIS, *Homotopy theory of infinite dimensional manifolds*, Topology, 5 (1966), 1-16.

[35] W. V. PETRYSHYN, *On the approximation-solvability of nonlinear equations*, Math. Annalen., 177 (1968), 156-164.

[36] B. N. SADOVSKII, *On a fixed point principle*, Functional Analysis and Appl., 1 (1967), 74-76.

[37] R. B. THOMPSON, *A unified approach to local and global fixed point indices*, Advances in Math., 3 (1969), 1-72.

[38] M. FURI and A. VIGNOLI, *A Fixed point theorem in complete metric spaces*, Boll. Unione Matem. Ital., serie IV (N. 4-5), (1969), 505-506.

[39] — —, *On α-nonexpansive mappings and fixed points*, Rend. Acc. Naz. Lincei 48 (N. 2). (1970).

[40] — —, *On a property of the unit sphere in a linear normed space*, Boll. Acad. Pol. Sci., 18 (2), (1970), 115-116.

REFERENCES

[1] P. ALEXANDROFF and H. HOPF, Topologie, Berlin, 1935.

[2] A. AMBROSETTI, *Un teorema di esistenza per le equazioni differenziali sugli spazi di Banach,* Rend. Sem. Mat. Univ. Padua, 39 (1967), 349-361.

[3] JU. G. BORISOVIC and JU. I. SAPRONOV, *A contribution to the topological theory of condensing operators,* Soviet Math. Dokladi, 9 (1968), 1304-1307.

[4] L. BROUWER, *Uber abbildungen vom mannigfaltigkeiton,* Math. Annalen., 71 (1912), 97-115.

[5] F. E. BROWDER, *The topological fixed point theory and its applications to functional analysis,* Unpublished Ph.D. Dissertation, Princeton University, 1948.

[6] — —, *On the fixed point index for continuous mappings of locally connected spaces,* Summa Brasil. Math., 4 (1960), 253-293.

[7] — —, *On the spectral theory of elliptic differential operators,* Math. Annalen., 142 (1961), 22-130.

[8] — —, *Asymptotic fixed point theorems,* Math. Annalen., 185 (1970), 38-61.

[9] — —, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Proceedings of the Symposium on Nonlinear Functional Analysis, Amer. Math. Soc., Chicago, April, 1968 (to appear).

[10] — — and R. D. NUSSBAUM, *The topological degree for noncompact nonlinear mappings in Banach spaces,* Bull. Amer. Math., Soc., 74 (1968), 641-646.

[11] — — and W. V. PETRYSHYN, *Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces,* Jour. of Functional Analysis, 3 (1969), 217-245.

[12] G. DARBO, *Punti uniti in trasformazioni a codominio non compatto,* Rend. Sem. Mat. Univ. Padua, 24 (1955), 84-92.

[13] A. DELEANU, *Theorie des points fixes sur les retractes de voisinage des espaces convexoides,* Bull. Soc. Math. France, 89 (1959), 235-243.

[14] A. DOLD, *Fixed point index and fixed point theorems for Euclidean neighborhood retracts,* Topology, 4 (1965), 1-8.

[15] J. DUGUNDJI, *An extension of Tietze's theorem,* Pac. Jour. Math., 1 (1951), 353-367.

[16] D. G. de FIGUREIDO and L. A. KARLOVITZ, *On the radial projection in normed spaces,* Bull. Amer. Math. Soc., 73 (1966), 364-367.

[17] A. GRANAS, *Introduction to Topology of Functional Spaces,* University of Chicago Mathematics Lecture Notes, Spring, 1961.

[18] O. HANNER, *Some theorems on absolute neighborhood retracts,* Arkiv for Matematik, 1 (1951), 389-408.

[19] S. HU, *Theory of Retracts,* Detroit (Wayne State University Press), 1965.

[20] C. KURATOWSKI, *Sur les espaces complets,* Fund. Math., 15 (1930), 301-309.

[21] J. LERAY, *Sur la position d'un ensemble ferme de points d'un espace topologique,* Jour. de Math. Pures et Appl., 24 (1945), 169-199.

[22] — —, *Sur les equations et les transformations,* Jour. de Math. Pures et Appl., 24 (1945), 201-248.

[23] — —, *Theorie des points fixed, indice total, et nombre de Lefschetz,* Bull. Soc. Math. France, 87 (1959), 221-233.

PROOF. - It suffices to prove that F is an admissible homotopy. Since $F(x, t) \neq x$ for $(x, t) \in \partial G \times J$, $S = \{(x, t) \in G : F(x, t) = x\} = \{(x, t) \in \mathrm{cl}(G) : F(x, t) = x\}$ is closed and bounded.

The fact that $\gamma(F((A \cap G) \times J)) \leq \gamma(A)$ for any bounded subset A of X is immediate from our assumptions. We claim that $\Pi - F$ is a closed map, and to prove this it suffices to show $\Pi - F$ is a proper map. Thus let M be any compact subset of X and let $N = (\Pi - F)^{-1}(M) = \{(x, t) \in \mathrm{cl}(G) \times J : x - F(x, t) \in M\}$. Let $N_1 = \Pi(N)$. Since N is closed and $N \subset N_1 \times J$, it suffices to show $\gamma(N_1) = 0$. However, if $x \in N_1$, $x = F(x, t) + m$ for some $t \in J$ and some $m \in M$. Thus $N_1 \subset F(N_1 \times J) + M$. If $\gamma(N_1) > 0$, this would imply $\gamma(N_1) \leq \gamma(F(N_1 \times J) + M) = \gamma(F(N_1 \times J)) < \gamma(N_1)$, a contraction. Thus $\gamma(N_1) = 0$ and $\Pi - F$ is proper, S is compact and hence $F(S)$ is compact. Thus if $\{C_k : k \in K\}$ is a locally finite covering of X by closed, convex sets, there exists an open neighborhood W of $F(S)$ such that $\mathrm{cl}(W) \cap C_k$ is empty, except for finitely many k. Setting $U = F^{-1}(W)$ it is clear that $\langle F, U, \{C_k : k \in K\}\rangle$ is a homotopy admissible triple. Q.E.D.

COROLLARY 2. - Let G be a bounded, open subset of $X \in \mathcal{F}$ and let $J = [0, 1]$. Let $F : \mathrm{cl}(G) \times J \to X$ be a continuous function such that $F(x, t) \neq x$ for $x \in \partial G \times J$. Assume that $F_t = F(\cdot, t)$ is a condensing map for $t \in [0, 1]$ and suppose that F is uniformly continuous in t. Then for any subset A of $\mathrm{cl}(G)$ with $\gamma(A) > 0$, $\gamma(F(A \times J)) < \gamma(A)$, so that by Corollary 1, $i_X(F_0, G) = i_X(F_1, G)$.

PROOF. - Assume $A \subset \mathrm{cl}(G)$ and $\gamma(A) > 0$. Our first claim is that $g(t) = \gamma(F_t(A))$ is a continuous function. To see this, select $t_0 \in J$ and take $\varepsilon > 0$. By the uniform continuity of F_t, select $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|F_t(x) - F_{t_0}(x)\| < \varepsilon/2$ for all $x \in A$. Therefore, for $|t - t_0| < \delta$, $F_t(A) \subset N_{\varepsilon/2}(F_{t_0}(A))$ and $F_{t_0}(A) \subset N_{\varepsilon/2}(F_t(A))$. The first inclusion implies $g(t) \leq g(t_0) + \varepsilon$ and the second that $g(t_0) - \varepsilon \leq g(t)$. Thus $h(s) = g(s)/\gamma(A)$ is continuous; and since $h(s) < 1$ for $s \in [0, 1]$, $h(s) \leq k \leq 1$ for $s \in [0, 1]$. It now follows just as in the proof of Proposition 1, Section E, that $\gamma(F(A \times J)) \leq k\gamma(A) < \gamma(A)$.

Q.E.D.

We shall have no need for further generalizations of commutativity and normalization properties of Section E, and we shall consequently limit ourselves to the results already stated. We should mention that at least the commutativity property of E easily generalizes to the context of this section.

REMARK. - After this paper was written A. VIGNOLI kindly brought to our attention a number of papers in this area which he has written in collaboration with M. FURI. In particular it seems that the notion of condensing maps was first introduced by FURI and VIGNOLI in [38] under the name « densifying maps ». SADOVSKII cannot be given credit since (as we have already noted) he used a different measure of noncompactness from the one used here.

sequence of admissible approximations h_n with respect to $<g, U, \{C_j : j \in J\}>$ such that $\sup\{\|g(x) - h_n(x)\| : x \in cl(U)\} \to 0$ and $i_X(h_n, U) = i_X(h, U) \neq 0$. Thus h_n has a fixed point $x_n \in cl(U)$ and $x_n - g(x_n) \to 0$. Since $I - g | cl(U)$ is a closed map, g has a fixed point in $cl(U)$. Q.E.D.

Before proceeding further we need to define the concept of an *admissible homotopy for the fixed point index*. Let $J = [0, 1]$, the unit interval, and let Ω be an open subset of $X \times J$, $X \in \mathfrak{F}$. We shall say that a continuous map $F : \Omega \to X$ is an *admissible homotopy* if (1) $S \equiv \{(x, t) \in \Omega : F(x, t) = x\}$ is closed and bounded. (2) There exists a bounded open neighborhood U of S with $cl(U) \subset \Omega$ and a locally finite covering $\{C_k : k \in K\}$ of X by closed, convex sets $C_k \subset X$ such that the following properties hold: (a) For any bounded subset A of X, $\gamma(F((A \times J) \cap cl(U))) \leq \gamma(A)$. (b) $\Pi - F | cl(U)$ is a closed map, where Π is the projection $\Pi(x, t) = x$. (c) $F(cl(U)) \cap C_k$ is empty except for finitely many k. If F, U, and $\{C_k : k \in K\}$ are as above we say $<F, U, \{C_k : k \in K\}>$ is a *homotopy admissible triple.*

THEOREM 3. (Homotopy property). - Let Ω be an open subset of $X \times J$, $X \in \mathfrak{F}$, $J = [0, 1]$, and let $F : \Omega \to X$ be admissible. Then if we write $\Omega_t = \{x : (x, t) \in \Omega\}$ and $F_t = F(\cdot, t)$, $F_t : \Omega_t \to X$ is an admissible map and $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

PROOF. - If $<F, U, \{C_k : k \in K\}>$ is a homotopy admissible triple, it is clear that $<F_t, U_t, \{C_k : k \in K\}>$ is an admissible triple ($U_t = \{x : (x, t) \in U\}$), so $F_t : \Omega_t \to Y$ is an admissible map. Let $\delta = \inf\{\|(\Pi - F)(x, t)\| : (x, t) \in cl(U) \sim U\}$ and notice that by our hypotheses, $\delta > 0$. By assumption $F(clU) \cap C_k$ is empty for $k \notin L$, L some finite subset of K. Let $C = \cup_{k \in L} C_k$ and let $R : C \to D$ be a retraction of C onto a compact subset D of C such that $R(x) \in C_k$ if $x \in C_k$. (R exists by Corollary 1, Section B). It is easy to show that there exists M such that $\|F(x, t)\| \leq M$ for $(x, t) \in cl(U)$ and $\|y\| \leq M$ for $y \in D$. Select s such that $1 - \frac{\delta}{2M} < s < 1$ and define $H(x, t) = sF(x, t)) + (1 - s)R(x)$. It is easy to see that $\|H(x, t) - F(x, t)\| < \delta$ for $(x, t) \in cl(U)$, so $H(x, t) \neq x$ for $x \in cl(U) \sim U$. Our construction also implies that if $F(x, t) \in C_k$, then $H(x, t) \in C_k$. Finally, for any subset A of X, $\gamma(H(A \times J) \cap clU)) \leq s\gamma(F((A \times J) \cap clU)) \leq s\gamma(A)$, since R is compact. We now apply Corollary 1, Section E, and setting $H_t = H(\cdot, t)$, we find $i_X(H_0, U_0) = i_X(H_1, U_1)$. However, it is clear that H_t is an admissible approximation with respect to $<F_t, U_t, \{C_k . k \in K\}>$, so $i_X(H_t, U_t) = i_X(F_t, \Omega_t)$. Thus we have $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$. Q.E.D.

COROLLARY 1. - Let G be a bounded open subset of $X \in \mathfrak{F}$ and let $J = [0, 1]$. Let $F : cl(G) \times J \to K$ be a continuous map such that $F(x, t) \neq x$ for $x \in \partial G$, $t \in J$. Assume that for any subset A of $cl(G)$ such that $\gamma(A) \neq 0$, $\gamma(F(A \times J)) < \gamma(A)$. Then $i_X(F_0, G) = i_X(F_1, G)$.

$<g, V, \{D_l : l \in L\}>$ such that $\|h'(x) - g(x)\| < \eta$ for $x \in \mathrm{cl}(V)$. Consider the homotopy $h_s = sh + (1-s)h'$, $0 \leq s \leq 1$. Since $\|h(x) - g(x)\|$ and $\|h'(x) - g(x)\|$ are strictly less than $\|x - g(x)\|$ for $x \in \partial V$, $h_s(x) \neq x$ for $x \in \partial V$. Since $h(x)$ and $h'(x)$ both lie in $C_j \ni g(x)$, $h_s(x) \in X$. Thus the homotopy is permissible and $i_X(h, V) = i_X(h', V)$. However, $h'(x) \neq x$ for $x \in V_1$, so $i_X(h', U \cap V) = i_X(h', V)$. Similarly, $i_X(f, U) = i_X f', U \cap V)$, where f' is an admissible approximation with respect to $<g, U \cap V, \{C_j : j \in J\}>$. Thus from the start we may as well assume that f and h are admissible approximations with respect to $<g, U \cap V, \{C_j : j \in J\}>$ and $<g, U \cap V, \{D_l : l \in L\}>$ respectively and try to prove $i_X(f, U \cap V) = i_X(h, U \cap V)$.

Consider the admissible triple $<g, U \cap V, \{C_j \cap D_l : (j, l) \in J \times L\}>$ and and let θ be an admissible approximation with respect to this triple. We claim that $i_X(f, U \cap V) = i_X(\theta, U \cap V) = i_X(h, U \cap V)$. It suffices to prove the first equality, the proof of the other being the same. Consider the homotopy $\theta_s(x) = (1-s)\theta(x) + sf(x)$, $0 \leq s \leq 1$. Just as above, $\theta_s(x) \neq x$ for $x \in \partial(U \cap V)$, $0 \leq s \leq 1$. Further, $\theta_s(x) \in X$ for $x \in \mathrm{cl}(U \cap V)$ and $0 \leq s \leq 1$. For if $g(x) \in C_j \cap D_l$, $\theta(x) \in C_j \cap D_l \subset C_j$ and $f(x) \in C_j$, so $\theta_s(x) \in C_j$. Thus the homotopy is permissible and $i_X(f, U \cap V) = i_X(\theta, U \cap V)$. Q.E.D.

DEFINITION. - Let G be an open subset of a space $X \in \mathfrak{F}$ and let $g : G \to X$ be a continuous function which is admissible. Let $<g, U, \{C_j : j \in J\}>$ be an admissible triple and let f be an admissible approximation with respect to this triple. We define $i_X(g, G) = i_X(f, U)$. Theorem 1 shows this definition is well-defined.

THEOREM 1 (Additivity property). - Let G be an open subset of a space $X \in \mathfrak{F}$ and $g : G \to X$ be an admissible map. Let $S = \{x \in G \mid g(x) = x\}$ and assume $S \subset G_1 \cup G_2$, G_1 and G_2 disjoint open subsets of G. Then g is admissible as a map from G_i to X ($i = 1, 2$) and $i_X(g, G) = i_X(g, G_1) + i_X(g, G_2)$. Further, if $i_X(g, G) \neq 0$, then g has a fixed point in G.

PROOF. - Since g is an admissible map, let $<g, U, \{C_j : j \in J\}>$ be an admissible triple. Let $S = \{x \in G : g(x) = x\}$ and let $S_i = S \cap G_i$, $1 \leq i \leq 2$. S_i is bounded since S is bounded and S_i is closed since $S_1 = S \cap G_2'$ and $S_2 = S \cap G_1'$, $G_i' =$ complement of G. Let U_i be an open neighborhood of S_i such that $\mathrm{cl}(U_i) \subset G_i \cap U$. It is clear that $<g \mid G_i, U_i, \{C_j : j \in J\}>$, $i = 1, 2$, are admissible triples. Let $\delta = \inf\{\|x - g(x)\| : x \in \mathrm{cl}(U) \sim (U_1 \cup U_2)\}$ and let h be an admissible approximation with respect to $<g, U, \{C_j : j \in J\}>$ such that $\|h(x) - g(x)\| < \delta$ for $x \in \mathrm{cl}(U)$. Then we have $i_X(g, G) = i_X(h, U) = i_X(h, U_1) + i_X(h, U_2) = i_X(g, G_1) + i_X(g, G_2)$.

If $i_X(g, G) \neq 0$, there must exist an admissible triple $<g, U, \{C_j : j \in J\}>$ such that U is nonempty and an admissible approximation h with respect to this admissible triple, such that $i_X(h, U) \neq 0$. By Theorem 1, there exists a locally finite covering of X, $\{C_j : j \in J\}$, by closed, convex sets $C_j \subset X$. Since $S = g(S)$ is compact and the covering is locally finite, there exists an open neighborhood 0 of $g(S)$ such that $\text{cl}(0) \cap C_j$ is empty for finitely many j. Setting $W = g^{-1}(0)$, $g(\text{cl}(W)) \cap C_j$ is empty except for finitely many j. If we let $U = V \cap W$, it is easy to see that $< g, U, \{C_j : j \in J\} >$ is an admissible triple and g is an admissible map.

The last statement of the theorem is clear since by Lemma 1, $\{x \in \text{cl}(G) : g(x) = x\}$ is compact and $\{x \in \text{cl}(G) : g(x) = x\} = S$. Q.E.D.

Now let g be an admissible map as in the first paragraph and let $< g, U, \{C_j : j \in J\} >$ be an admissible triple. Since $(I - g)(x) \neq 0$ for $x \in \partial U$ and since $(I - g)|\text{cl}(U)$ is closed map, $\inf \{\|(I - g)(x)\| : x \in \partial U\} = \delta > 0$. If $f : \text{cl}(U) \to X$ is a continuous map, we shall say that f is an *admissible approximation with respect to* $< g, U, \{C_j : j \in J\} >$ if (1) f is a k-set-contraction, $k < 1$.

(2) $\|f(x) - g(x)\| < \delta$ for $x \in \text{cl}(U)$, $\delta = \inf\{\|(I - g)(x)\| : x \in \partial U\}$.

(3) For all $j \in J$ and $x \in \text{cl}(U)$, if $g(x) \in C_j$, then $f(x) \in C_j$.

THEOREM 1. - Let G be an open subset of $X \in \mathfrak{F}$, $g : G \to X$ an admissible map. Let $< g, U, \{C_j : j \in J\} >$ be an admissible triple. Then given $\eta > 0$, there exists an admissible approximation f with respect to $< g, U, \{C_j : j \in J\} >$ such that $\|g(x) - f(x)\| < \eta$ for $x \in \text{cl}(U)$. Furthermore if $< g, V, \{D_l : l \in L\} >$ is another admissible triple and h is an admissible approximation with respect to $< g, V, \{D_l : l \in L\} >$, then $i_X(h, V) = i_X(f, U)$. (Notice that the assumption that h and f are admissible approximations guarantes that $i_X(h, V)$ and $i_X(f, U)$ are defined).

PROOF. - First we show that admissible approximations exist. By assumption $g(\text{cl}(U)) \cap C_j$ is empty unless $j \in F$, F some finite subset of J. Consider $C = \cup_{j \in F} C_j$. By Corollary 1, $I - B$, there exists a retraction $R : C \to K$, where K is some compact subset of C, such that $R(y) \in C_j$ if $y \in C_j$ for all $y \in C$, $j \in F$. Since $g|\text{cl}(U)$ is a 1-set-contraction and $\text{cl}(U)$ is bounded, $\|g(x)\| \leq M$ for $x \in \text{cl}(U)$; and since K is compact, we can also assume $\|y\| \leq M$ for $y \in K$. Let $\delta = \inf\{\|x - g(x)\| : x \in \partial U\}$ and for $0 < \eta < \delta$ and $1 - \eta/2M < < t < 1$, define $f(x) = tg(x) + (1 - t)R(g(x))$. From our construction it is clear that $\|f(x) - g(x)\| < \eta$ for $x \in \text{cl}(U)$. Since g is a 1-set-contraction and R is compact, f is a t-set-contraction, $t < 1$. Finally, since $R(y) \in C_j$ if $y \in C_j$ for $j \in F$ and since C_j is convex, $f(x) \in C_j$ if $g(x) \in C_j$.

We next have to show that $i_X(h, V) = i_X(f, U)$. Since $I - g|\text{cl}(V)$ is a closed map and $(I - g)(x) \neq 0$ for $\in \text{cl}(V) - U \cap V = V_1$, $\inf\{\|(I - g)(x)\| : x \in V_1\} = = \eta > 0$. (Notice that $U \cap V$ may be empty if S is empty). Using the first part of this theorem, let h' be an admissible approximation with respect to

F. - The fixed point index for 1-set-contractions.

In this section we shall define a fixed point index for maps which are essentially 1-set-contractions and which satisfy certain additional conditions. Our main interest will be in a corollary of this work, the fixed point index for condensing maps, but we would save no effort by initially restricting ourselves to that case. Specifically, we consider the following situation: Suppose $X \in \mathcal{F}$, G is an open subset of X, and $g: G \to X$ is a continuous map. We shall say that g is an *admissible map for the fixed point index* or simply an *admissible map* iff

(1) $S = \{x \in G : g(x) = x\}$ is closed and bounded.

(2) There exists a bounded, open neighborhood U of S with $\mathrm{cl}(U) \subset G$ and a locally finite covering $\{C_j : j \in J\}$ of X by closed, convex sets $C_j \subset X$ such that (*a*) $g|\mathrm{cl}(U)$ is a 1-set-contraction, (*b*) $I - g|\mathrm{cl}(U)$ is a closed map ($I =$ the identity on the BANACH space B containing X), and (*c*) $g(\mathrm{cl}(U)) \cap C_j$ is empty except for finitely many $j \in J$. If S is empty, U may be empty. If g, U, and $\{C_j : j \in J\}$ are as above, we shall say that $< g, U, \{C_j : j \in J\} >$ is an *admissible triple*.

Before proceding further let us consider some classes of admissible maps. If G is a bounded, open subset of a BANACH space X and $g : \mathrm{cl}(G) \to X$ is a 1-set-contraction such that $g(x) \neq x$ for $x \in \partial G$ and $I - g|\mathrm{cl}(G)$ is a closed map, then g is an admissible map. To give another example we need the following lemma.

LEMMA 1. - Let A be a closed, bounded subset of a BANACH space B. Let $g : A \to B$ be a condensing map. Then $(I - g)$ is a proper map.

PROOF. - Let M be a compact subset of B. We must show that $N = \{x \in A : (I - g)(x) \in M\}$ is compact. Clearly N is closed, so it suffices to show $\gamma(N) = 0$. If $x \in N$, $x = g(x) + m$ for some $m \in M$, and thus $N \subset g(N) + M$. If $\gamma(N) > 0$, $\gamma(N) \leq \gamma(g(N)) + \gamma(M) = \gamma(g(N)) < \gamma(N)$, a contradiction. Q.E.D.

PROPOSITION 1. - Let $X \in \mathcal{F}$ and let G be an open subset of $\mathcal{F}$. Let $g : G \to X$ be a local condensing map such that $S = \{x \in G : g(x) = x\}$ is compact. Then g is an admissible map. In particular, if G is bounded, and $g : \mathrm{cl}(G) \to X$ is a condensing map such that $g(x) \neq x$ for $x \in \partial G$, then S is compact, so that g is admissible in that case.

PROOF. - Since S is compact and g is a local condensing map, there exists an open neighborhood V of S such that $\mathrm{cl}(V) \subset G$ and $g|\mathrm{cl}(V)$ is a condensing map. Clearly $g|\mathrm{cl}(V)$ is a 1-set-contraction, and by Lemma 1, $I - g|\mathrm{cl}(V)$ is a proper map and hence closed. Since $X \in \mathcal{F}$, there exists a

PROOF. - Let us set $K^*_\infty = \cap_{m\geq 1} K^*_m$. We then have $\mathrm{cl}(G) \cap K^*_\infty = \cap_{m\geq 1} \mathrm{cl}(G) \cap K^*_m$, and since $\gamma(\mathrm{cl}(G) \cap K^*_m) \to 0$, Proposition 2 of $I - A$ shows that given any open neighborhood V of $\mathrm{cl}(G) \cap K^*_\infty$, $\mathrm{cl}(G) \cap K^*_m \subset V$ for $m \geq m_1$. Since $f(\mathrm{cl}(G) \cap K^*_\infty)$ is a compact subset of G, we can find $\delta > 0$ such that $N_{2\delta}(f(\mathrm{cl}(G) \cap K_\infty)) \subset G$. Setting $V = f^{-1}(N_\delta(f(\mathrm{cl}(G) \cap K^*_\infty)))$, which is an open neighborhood of $\mathrm{cl}(G) \cap K^*_\infty$, for $m \geq m_1$ we have $\mathrm{cl}(G) \cap K^*_m \subset V$, so $N_\delta(f(\mathrm{cl}(G \cap K^*_m))) \subset N_\delta(f(\mathrm{cl}(G) \cap K^*_m)) \subset N_{2\delta}(f(\mathrm{cl}(G) \cap K^*_\infty)) \subset G$. Q.E.D.

We can now prove Theorem 4. Select δ and m_1 as in Lemma 7. Suppose that $X = \cup_{i\in J} C_i$ is a locally finite union of closed, comvex sets in B. Since K^*_∞ is a compact subset of X, let W be an open neighborhood of K^*_∞ such that $W \cap C_i = \emptyset$ unless $i \in F$, where F is some finite subset of J. Select m_2 such that for $m \geq m_2$, $K^*_m \subset W$, as we can do, since $\gamma(K^*_m) \to 0$. It follows that for $m \geq m_2$, $K^*_m = \cup_{i\in F}(\overline{\mathrm{co}}\, f(G \cap K^*_{m-1})) \cap C_i$, a finite union of closed, convex sets. It is clear that Corollary 2, Section B, applies here, so for δ as above and $m \geq m_3$, we can find a deformation retraction $H_m : K^*_m \times I \to K^*_m$, $H_m(x, t) = x$ for $x \in K$, $t \in {}^*_\infty I$, $H_m(\cdot, 1)$ a retraction onto K_∞, and $\|H_m(x, t) - x\| < \delta$ for $x \in K^*_m$, $t \in I$. For $m \geq \max\{m_1, m_3\}$ and $x \in \mathrm{cl}(G \cap K^*_m)$, consider the homotopy $F_m(x, t) = H_m(f(x), t)$, $0 \leq t \leq 1$. It is clear that $F_m(x, t) \in K^*_m$, and since $\|H_m(f(x), t) - f(x)\| < \delta$ and $N_\delta(f(\mathrm{cl}(G \cap K^*_m)) \subset G$, $H_m(f(x), t) \in G$. It follows that $F_m : (G \cap K^*_m) \times I \to G \cap K^*_m$, so the maps $f_t : G \cap K^*_m \to G \cap K^*_m$ defined by $f_t(x) = F_m(x, t)$ induce the same map in homology; and if we write $r_m = H_m(\cdot, 1)$, which is a retraction of K^*_m onto K^*_∞, $(r_m f) : G \cap K^*_m \to G \cap K$ and $f : G \cap K^*_m \to$ $\to G \cap K^*_m$ induce the same maps in homology. It follows that $\Lambda_{\mathrm{gen}}(f | G \cap K^*_m) =$ $= \Lambda_{\mathrm{gen}}(r_m f | G \cap K^*_m)$, so it suffices to show that $\Lambda_{\mathrm{gen}}(r_m f | G \cap K^*_m)$ is defined and equal to $i_X(f, X)$. However, $(r_m f)(G \cap K^*_m) \subset G \cap K^*_\infty$, so by Lemma 4 we obtain $\Lambda_{\mathrm{gen}}(r_m f | G \cap K^*_m) = \Lambda_{\mathrm{gen}}(r_m f | G \cap K^*_\infty) = \Lambda_{\mathrm{gen}}(f | G \cap K^*_\infty)$.

We are almost done. By the additivity property, $i_X(f, X) = i_X(f, G) \equiv$ $\equiv i_{K^*_\infty}(f, G \cap K_\infty)$. Since $\mathrm{cl}(f(G \cap K^*_\infty))$ is a compact subset of $G \cap K^*_\infty$, we can find an open neighborhood U in $G \cap K^*_\infty$ of $\mathrm{cl}(f(G \cap K^*_\infty))$ such that $\mathrm{cl}(U)$ is a compact subset of $G \cap K^*_\infty$. Cover $\mathrm{cl}(U)$ by a finite number of compact, convex sets $D_i \subset G \cap K^*_\infty$, $1 \leq i \leq n$, and let $A = \cup_{i=1}^n D_i$. Then A is a compact ANR, $f(A) \subset A$, and in fact $f(G \cap K^*_\infty) \subset A$. By Lemma 3, we have $\Lambda_{\mathrm{gen}}(f | G \cap K^*_\infty) = \Lambda_{\mathrm{gen}}(f | A)$, which we know is defined. On the other hand, since all fixed points of f in $G \cap K^*$ lie in U, $i_{K^*_\infty}(f, G \cap K^*) = i_{K^*_\infty}(f, U)$. Furthermore, by the normalization property, $\Lambda_{\mathrm{gen}}(f | A) = \Lambda(f | A) = i_A(f, A) =$ $= i_A(f, U) = i_{K^*_\infty}(f, U)$, since $f(U) \subset A$ and $U \cap A = U$. This shows that $\Lambda_{\mathrm{gen}}(f | A) = i_X(f, X)$. Q.E.D.

since $ST(F) = ST(S(E)) = S(TS(E)) \subset S(E) = F$. If $w \in W$, by definition of E there exists an m such that $(TS)^m(Tw) \in E$, so $(ST)^{m+1}(w) = S(TS)^m(Tw) \in S(E) = F$. Thus $tr_{\text{gen}}(ST)$ is defined and $tr_{\text{gen}}(ST) = tr(ST|F)$. However, we have $E \underset{S}{\rightarrow} F \underset{T}{\rightarrow} E$, so the usual commutativity property implies $tr_{\text{gen}}(TS) = tr(TS|E) = tr(ST|F) = tr_{\text{gen}}(ST)$. Q.E.D.

LEMMA 4. - Let X be a topological space and $f: X \rightarrow X$ be a continuous map. Let Y be an subspace of X such that f maps Y into itself and $Y \supset f(X)$. Then $\Lambda_{\text{gen}}(f)$ is defined if and only $\Lambda_{\text{gen}}(f|Y)$ is defined, and $\Lambda_{\text{gen}}(f) = \Lambda_{\text{gen}}(f|Y)$. $f|Y$ is meant to indicate a map from Y to Y.

PROOF. - Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle i} & \\ & Y & \end{array}$$

where i denotes inclusion and g denotes f viewed as a map from X to Y. We have $f_{*,j} = i_{*,j}g_{*,j}: H_j(X) \rightarrow H_j(X)$. However $g_{*,j}: H_j(X) \rightarrow H_j(Y)$ and $i_{*,j}: H_j(Y) \rightarrow H_j(X)$, so Lemma 3 implies that $tr_{\text{gen}}(f_{*,j}) = tr_{\text{gen}}(g_{*,j}i_{*,j})$. Since $g_{*,j}i_{*,j} = (f|Y)_{*,j}$, $tr_{\text{gen}}(f_{*,j}) = tr_{\text{gen}}((f|Y)_{*,j})$, and it follows that $\Lambda_{\text{gen}}(f) = \Lambda_{\text{gen}}(f|Y)$. Q.E.D.

LEMMA 5. - Let X be a topological space and $f: X \rightarrow X$ a continuous map. Let G be a subspace of X and suppose $f: G \rightarrow G$ and $f^n(X) \subset G$ for some $n \geq 1$. Then $\Lambda_{\text{gen}}(f)$ is defined if and only if $\Lambda_{\text{gen}}(f|G)$ is defined and $\Lambda_{\text{gen}}(f) = \Lambda_{\text{gen}}(f|G)$.

PROOF. - Let $G_i = f^i(X) \cup G$, $1 \leq i \leq n$, and since $f(G_i) \subset G_i$, write $f|G_i$, which we will view as a map from G_i to G_i. Applying Lemma 3 repeatedly, we find that $\Lambda_{\text{gen}}(f) = \Lambda_{\text{gen}}(f|G_1) = \ldots = \Lambda_{\text{gen}}(f|G_n) = \Lambda_{\text{gen}}(f|G)$. Q.E.D.

Up until now our lemmas have been very general. Our next lemma makes use of the hypotheses of Theorem 4. We shall not prove Lemma 6 here, since it appears more naturally in a sequence of results in [32].

LEMMA 6, - Under the hypotheses of Theorem 4, there exists a bounded open neighborhood G of $f^n(X)$ such that $\text{cl}(f(G)) \subset G$.

Now let G be as in Lemma 6 and let $K_m^* = \overline{\text{co}}\, f(G \cap K_{m-1}^*) \cap X$ and $K_1^* = \overline{\text{co}}\, f(G) \cap X$. If we write $G_m = G \cap K_m^*$ and $G_0 = G$, we see that $f: G_m \rightarrow G_{m+1}$ and $G_m \supset G_{m+1}$, so that by Lemma 3 we have $\Lambda_{\text{gen}}(f|G)$ is defined iff $\Lambda_{\text{gen}}(f|G_m)$ is defined ($f|G_m$ is viewed as a map from G_m to G_m) and $\Lambda_{\text{gen}}(f|G) = \Lambda_{\text{gen}}(f|G_m)$. Thus in order to prove Theorem 4, it suffices to show that $\Lambda_{\text{gen}}(f|G \cap K_m^*)$ is defined and equal to $i_X(f, X)$ for m large enough.

LEMMA 7. - We can find $\delta > 0$ and m_1 such that for $m \geq m_1$, $N_\delta(f(\text{cl}(G \cap K_m^*))) \subset G$.

Consider V/N, where N is as in definition B. It is easy to see that $\theta(N)=0$, so we obtain $\bar{\theta}: V/N \to F$. To see that $\bar{\theta}$ is an isomorphism, it suffices to show that $\theta^{-1}(0)=N$. Take $x \in V - N$ and select m so that $S^m(x) \in F$. Since $x \notin N$, $S^m(x) \neq 0$; and since $S|F$ is an isomorphism, if we take $z \in F$ such that $S^m(z) = S^m(x) \neq 0$, $z \neq 0$, i.e., $\theta(x) \neq 0$. Thus we see that $\bar{\theta}: V/N \to S^n(E)$ is an isomorphism.

Finally, notice that we have commutativity in the following diagram:

$$\begin{array}{ccc} V/N & \xrightarrow{\ \bar{\theta}\ } & F \\ \uparrow {\scriptstyle \bar{S}} & & \uparrow {\scriptstyle S} \\ V/N & \xrightarrow{\ \bar{\theta}\ } & F \end{array}$$

It follows that $tr(\bar{S}) = tr(\bar{\theta}^{-1}(S|F(\bar{\theta})) = tr(S|F) =$ BROWDER's generalized trace.

Conversely, let us show that $B \Rightarrow A$. Suppose that V/N is finite dimensional and let $[v_1], \ldots, [v_n]$ be a basis; $[v]$ denotes the equivalence class of $v \in V$ in V/N. By definition, $Sv_i - \sum_{j=1}^n a_{ij}v_j \in N$ for some a_{ij}, $1 \leq j \leq n$, so $S^{m_i}(Sv_i - \sum_{j=1}^n a_{ij}v_j) = 0$ and $S^{m_i+1}v_i = \sum_{j=1}^n a_{ij}S^{m_i}v_j$. Select $m \geq \max\{m_1, \ldots, m_n\}$ and consider the subspace E spanned by $S^j v_i$, $1 \leq i \leq n$, $0 \leq j \leq m$. Since $S^{m+1}v_i = \sum_{j=1}^n a_{ij}S^m v_j$, S maps E into itself. Given $v \in V$, by definition $v - \sum_{j=1}^n b_i v_i \in N$ for some b_i, $1 \leq i \leq n$; and we find $S^k v = \sum_{i=1}^n b_i S^k v_i \in E$ for some $k \geq 0$. It follows that E meets the conditions of definition A. Now just use the proof that $A \Rightarrow B$ to show $tr(\bar{S}) = tr(S|E)$, where $\bar{S}: V/N \to V/N$.

Q.E.D.

If X is a topological space, $f: X \to X$ a continuous map, $H_{*,i}(X)$ (coefficients in the rationals) is a vector space and $f_{*,i}$ a linear map. We define $\Lambda_{\text{gen}}(f) = \sum_{i\geq 0}(-1)^i tr_{\text{gen}}(f_{*,i})$, where we assume $tr_{\text{gen}}(f_{q,i})$ is defined for all i and zero except for finitely many i. The above lemma shows we can use either LERAY's or BROWDER's definition of the generalized trace. We shall use BROWDER's definition. Our next lemma is due to LERAY, we prove if for completeness.

LEMMA 3. - Suppose we have $V \xrightarrow{S} W \xrightarrow{T} V$, where V and W are vector spaces. S and T are linear maps. If $tr_{\text{gen}}(TS)$ is defined, $tr_{\text{gen}}(ST)$ is defined and $tr_{\text{gen}}(TS) = tr_{\text{gen}}(ST)$.

PROOF. - Assume $tr_{\text{gen}}(TS)$ is defined and select a finite dimensional subspace E of V which meets the conditions of definition A for the linear map TS. Let $F = S(E) \subset W$. Clearly, F is finite dimensional, and $ST: F \to F$

In the general case, recall that $K_1 = \overline{\text{co}}\, g_1(G_1)$, $K_2 = \overline{\text{co}}\, g_2(G_1 \cap K_1)$ and generally, $K_{2n-1} = \overline{\text{co}}(G_1 \cap K_{2n-2})$ and $K_{2n} = \overline{\text{co}}\, g_2(G_2 \cap K_{2n-1})$. Thus we have $\gamma(K_1) \leq k_1\gamma(G_1)$, $\gamma(K_2) \leq k_2 k_1 \gamma(G_1)$ and, generally, $\gamma(K_{2n-1}) \leq k_1(k_1k_2)^{n-1}\gamma(G_1)$ and $\gamma(K_{2n}) \leq (k_1k_2)^n\gamma(G_1)$. Since $k_1k_2 < 1$, $\gamma(K_{2n-1})$ and $\gamma(K_{2n})$ opproach zero, so K_{odd} and K_{even} are compact. Q.E.D.

The reader may have noticed that while we have generalized the additivity, commutativity, and homotopy properties for the ordinary fixed point index to the context of k-set-contractions, we have avoided any attempts at generalizing the normalization property. We shall now remedy this omission. Our goal is to establish the following theorem.

THEOREM 4. - (Normalization property). Let $X \in \mathfrak{F}$ be a metric ANR and suppose that $X \subset B$, B a BANACH space, and X inherits its metric from the norm on B. Suppose that $f: X \to X$ is a k-set-contraction, $k < 1$, and assume that $f^n(X)$ is bounded for some $n \geq 1$. Then $\Lambda_{\text{gen}}(f)$, LERAY's generalized LEFSCHETZ number, is defined (using either singular or ČECH homology) and $\Lambda_{\text{gen}}(f) = i_X(f, X)$.

It is likely that Theorem 4 can be generalized, but the theorem as stated will suffice for our purposes.

In order to prove Theorem 4, we shall need a number of lemmas. Let us begin with some simple linear algebra and show that two definitions of a generalized trace, one due to BROWDER [8] and one to LERAY [23], are equivalent.

(A) BROWDER's definition: Let V be a vector space and $S: V \to V$ a linear map. Suppose there exists a finite dimensional subspace E such that $S: E \to E$ and such that for every $v \in V$, $S^m(v) \in E$ for some m depending on v. Define $tr_{\text{gen}}(S) = tr(S|E)$.

(B) LERAY's definition: Let V be a vector space and $S: V \to V$ a linear map. Let $N = \bigcup_{j=1}^{\infty} S^{-j}(0)$ and suppose V/N is finite dimensional. Since $S: N \to N$ we have $\bar{S}: V/N \to V/N$. Define $tr_{\text{gen}}(S) = tr(\bar{S})$.

LEMMA 2. - Definitions A and B are equivalent.

PROOF. - $A \Rightarrow B$. Suppose we have a finite dimensional subspace E as in definition A. Since E is finite dimensional, we can find n such that $S: S^n(E) \simeq S^{n+1}(E)$ is an isomosphism. Let us write $F = S^n(E)$. Given $v \in V$, $S^m(v) \in E$ from some m, so $S^{n+m}(v) = x \in F$. Since $S|F$ is an isomorphism, we can find $z \in F$ such that $S^{n+m}(z) = x = S^{n+m}(v)$. Let us define $\theta(v) = z$. We claim that $\theta(v)$ is well defined. For suppose $S^{n_1}(v) \in F$ and $S^{n_2}(v) \in F$, and we select z_1 and z_2 in F such that $S^{n_1}(z_1) = S^{n_1}(v)$ and $S^{n_2}(z_2) = S^{n_2}(v)$. We can assume that $n_2 \geq n_1$, so we have $S^{n_2-n_1}(S^{n_1}(z_1)) = S^{n_2}(z_1) = S^{n_2}(z_2)$. Since S is an isomorphism on F, $z_1 = z_2$, and θ is well defined. Clearly, θ is linear.

COROLLARY 5. - Let G be a bounded, open, convex set in a BANACH space X, $f: \mathrm{cl}(G) \to X$ a k-set-contraction, $k < 1$. Assume $f(\partial G) \subset \mathrm{cl}(G)$. Then f has a fixed point in $\mathrm{cl}(G)$.

PROOF. - Take $x_0 \in G$ and consider the homotopy $tf + (1-t)x_0$. If $f(x) \neq x$ for $x \in \partial G$, $tf(x) + (1-t)x_0 \neq x$, since $tf(x) + (1-t)x_0 \in G$ for $t \neq 1$. Thus in this case the homotopy is permissible and $i_X(f, G) = i_X(x_0, G)$. One easily shows thbt $i_X(x_0, G)$ is given by the LEFSCHETZ number of the identity map of a point to itself, so $i_X(x_0, G) = 1$ and f has a fixed point. If $f(x) = x$ for some $x \in \partial G$, of course we are done immediately. Q.E.D.

It is clear that if $0 \in G$ any open set and $f(x) \neq sx$ for $s \geq 1$, and $x \in \partial G$, then the same proof shows f has a fixed point.

Next, let us establish a commutativity property. The reader should note that in the proof of Theorem 3 below, we write γ generically to denote measure of noncompactness in two different metric spaces, X_1 and X_2. Of course the measures of noncompactness in these metric spaces actually are, in general, different.

THEOREM 3. - (The commutativity property). Let G_1 and G_2 be open subset of spaces X_1 and X_2, respectively, $X_i \in \mathfrak{F}$. Let $g_1 : G_1 \to X_2$ be a k_1-set-contraction and $g_2 : G_2 \to X_1$ be a k_2-set-contraction. Assume that $S_1 = \{x \in g_1^{-1}(G_2) \mid (g_2 g_1)(x) = x\}$ is compact. Finally assnme that $k_1 k_2 < 1$. If $k_1 = 0$ we only need assume g_2 is continuous and defined on $\mathrm{cl}(G_2)$. Then we have $i_{X_1}(g_2 g_1, g_1^{-1}(G_2)) = i_{X_2}(g_1 g_2, g_1^{-1}(G_1))$.

REMARK. - If $g_1^{-1}(G_2)$ is bounded, $(g_2 g_1)(x) \neq x$ for $x \in \mathrm{cl}(g_1^{-1}(G_2)) \sim g_1^{-1}(G_2)$, and $k_1 k_2 < 1$, then S_1 will be compact.

PROOF. - First we show that we can assume G_1 and G_2 bounded and g_i defined on $\mathrm{cl}(G_i)$. Since S_1 is compact, by Lemma 1, Section C, $S_2 = \{x \in g_2^{-1}(G_1) : (g_1 g_2)(x) = x\}$ is compact. Let $U_i \subset G_i$ be a bounded open neighborhood of S_i with $\mathrm{cl}(U_i) \subset G_i$ and let $H_1 = \{x \in U_1 : g_1(x) \in U_2\}$, $H_2 = \{x \in U_2 : g_2(x) \in U_1\}$. By Lemma 1, Section C, it is easy to see that H_i is an open neighborhood of S_i.

Thus $i_{X_1}(g_2 g_1, g_1^{-1}(G_2)) = i_{X_1}(g_2 g_1, H_1)$ and similarly for H_2. Since $H_1 = (g_1 \mid U_1)^{-1}(U_2)$ and $H_2 = (g_2 \mid U_2)^{-1}(U_1)$, we see that (since we can restrict attention to U_1 and U_2) we may as well assume G_1 and G_2 bounded and g_i defined on $\mathrm{cl}(G_i)$.

Thus assume G_1 and G_2 bounded and g_i defined on $\mathrm{cl}(G_i)$. In the notation of Theorem 3, Section C, it suffices to show K_{odd} and K_{even} are compact. If $k_1 = 0$, we know that $\mathrm{cl}\,(g_1(G_1))$ is compact, so $K_1 = \overline{\mathrm{co}}\, g_1(G_1)$ is compact. Also, we have $K_2 = \overline{\mathrm{co}}\, g_2(G_2 \cap K_1) = \overline{\mathrm{co}}\, g_2(\mathrm{cl} G_2 \cap K_1)$. But $\mathrm{cl} G_2 \cap K_1$ is compact and g_2 is continuous, so $g_2(\mathrm{cl} G_2 \cap K_1)$ is compact and K_2 is compact. Since $K_1 \supset K_{\mathrm{odd}}$ and $K_2 \supset K_{\mathrm{even}}$, K_{odd} and K_{even} are compact.

PROOF. - Suppose $B \subset A$ and $\gamma(B) = r$. We have to show that $F(B \times M)$ can be covered by a finite number of sets of diameter $\leq kr + \varepsilon$ for every $\varepsilon > 0$. Select $\varepsilon > 0$. For each $m \in M$ there is an open ball $B_{\delta(m)}(m)$ about m such that if $\rho(m', m) < \delta(m)$, $d(F_{m'}(x), F_m(x)) < \varepsilon/3$ for $x \in A$. Since M is compact, cover M by a finite number of ball $B_{\delta(m_i)}(m_i)$, $1 \leq i \leq n$. Since F_{m_i} is a k-set-contraction, there exists a covering of $F_{m_i}(B)$ by a finite number of sets S_{ij}, $1 \leq j \leq n_i$, of diameter $\leq kr + \varepsilon/3$. We know that $\gamma(N_{\varepsilon/3}(S_{ij})) \leq kr + \varepsilon$. On the other hand, suppose that $x \in B$ and $m \in M$. Then for some i, $\rho(m, m_i) < \delta(m_i)$, so that $d(F(x, m), F(x, m_i)) < \varepsilon/3$. But $F(x, m_i) \in S_{ij}$ for j, $1 \leq j \leq n_i$, so $F(x, m) \in N_{\varepsilon/3}(S_{ij})$. This shows $F(B \times M) \subset \bigcup_{i,j} N_{\varepsilon/3}(S_{ij})$, so that $\gamma(F(B \times M)) \leq kr + \varepsilon$.

If $S = \{x \in A : F(x, m) = x$ for some $m \in M\}$, since A is closed and M compact, it is clear that S is closed. Also, we see that $S \subset F(S \times M)$, so $\gamma(S) \leq \gamma(F(S \times M)) \leq k\gamma(S)$. It follows that $\gamma(S) = 0$, so S is compact.

Q.E.D.

COROLLARY 2. - Let A be a closed, bounded subset of a BANACH space X, $f : A \to X$ a k-set-contraction, $k < 1$. Let I denote the identity map. Then $I - f$ is a proper map, i.e., $(I - f)^{-1}$ (compact set) = compact set.

PROOF. - Let M be a compact subset of X. We want to show that $(I - f)^{-1}(M)$ is compact. Let $F(x, m) = f(x) + m$, $x \in A$, $m \in M$. Clearly the conditions of Proposition 1 are met, so $\{x : F(x) + m = x$ for some $m \in M\} = (I - f)^{-1}(M)$ is compact. Q.E.D.

Recally that a proper map from one metric space to another is closed (that is, takes closed sets to closed sets), so that $I - f$ is closed.

COROLLARY 3. - Let G be a bounded, open subset of a space X, $X \in \mathfrak{F}$. Let $J = [0, 1]$ and let $F : \mathrm{cl}(G) \times J \to X$ be a continuous map such that $F_t(x) \equiv F(x, t) \neq x$ for $x \in \mathrm{cl}(G) - G$. Assume that each F_t is k-set-contraction, $k < 1$, k independent of t. Finally assume that for each $t_0 \in J$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that for $|t - t_0| < \delta$, $t \in J$, and $x \in \mathrm{cl}(G)$, $\|F(x, t) - F(x, t_0)\| < \varepsilon$. Then we have $i_X(F_0, G) = i_X(F_1, G)$.

PROOF. - By Proposition 1, for $A \subset \mathrm{cl}(G)$, $\gamma(F(A \times J)) \leq k\gamma(A)$, so all the concitions of the homotopy property are met. Q.E.D.

COROLLARY 4. - Let f_0 and f_1 be, respectively, k_0 and k_1-set-contractions, $f_i : \mathrm{cl}(G) \to X$ where G and X are as in Corollary 2, $k_i < 1$. Let $F(x, t) = tf_0(x) + (1 - t)f_1(x)$, $t \in J = [0, 1]$. Then $\gamma(F(A \times J)) \leq k\gamma(A)$, $k = \max(k_0, k_1)$.

PROOF. - $F_t = tf_0 + (1 - t)f_1$ is a k-set-contraction, so it suffices to show that $t \to F_t$ is continuous. $f_0[\mathrm{cl}(G)]$ and $f_1[\mathrm{cl}(G)]$ are bounded sets, say bounded by M. Thus we have $\|tf_0(x) + (1 - t)f_1(x) - t_0f_0(x) - (1 - t_0)f_1(x)\| \leq |t - t_0| \|f_0(x)\| + |t - t_0| \|f_1(x)\| \leq 2|t - t_0| M$. Thus the conditions of Proposition 1 are met. Q.E.D.

THEOREM 2. - (The homotopy property). Let $I = [0, 1]$ and let Ω be an open subset of $X \times I$, $X \in \mathfrak{F}$. Let $F : \Omega \to X$ be a continuous function and assume that F is a local strict-set-contraction in the following sense: given $(x, t) \in \Omega$, there exists an open neighborhood of (x, t) in Ω, $N_{(x,t)}$, such that for any subset A of X, $\gamma(F(N_{(x,t)} \cap (A \times I))) \leq k_{(x,t)}\gamma(A)$, $k_{(x,t)} < 1$. Assume that $S = \{(x, t) \in \Omega : F(x, t) = x\}$ is compact. Then (in the notation of Theorem 2, Section D) $i_X(F_t, \Omega_t)$ is defined for $t \in I$ and $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

PROOF. - By Theorem 2, Section D, (using the notation of that theorem) it suffices to find an open neighborhood 0 of S, $0 \subset \Omega$, such that $K_\infty(F, 0)$ is compact. Because F is a local strict-set-contraction and because S is compact, there exist a finite open covering of S by bounded open sets N_i in Ω, $1 \leq i \leq n$, such that for $A \subset X$, $\gamma(F(N_i \cap (A \times I))) \leq k_i\gamma(A)$, $k_i < 1$. Set $0 = \bigcup_{i=1}^n N_i$ and $k = \max\{k_i\} < 1$. Just as in the proof of Lemma 1, for $A \subset X$ we find $\gamma(F(0 \cap (A \times I))) \leq k\gamma(A)$.

Recall that $K_1(F, 0) = \overline{\text{co}}\, F(0)$ and $K_{n+1}(F, 0) = \overline{\text{co}}\, F(0 \cap (K_n \times I))$, and for notational convenience set $K_n = K_n(F, 0)$ and $K_\infty = K_\infty(F, 0) = \bigcap_{n \geq 1} K_n$. To prove our theorem it suffices to show $\gamma(K_n) \to 0$, since then K_∞ will be compact. Since 0 is bounded, $F(0)$ is bounded and $\gamma(K_1) = M$ is defined. Generally $\gamma(K_{n+1}) = \gamma(\text{co}\, F(0 \cap (K_n \times I))) = \gamma(F(0 \cap (K_n \times I))) \leq k\gamma(K_n)$. This shows $\gamma(K_{n+1}) \leq$ $\leq k^n M \to 0$. Q.E.D.

COROLLARY 1. - Let $I = [0, 1]$ and let Ω be a bounded, open subset of $X \times I$, $X \in \mathfrak{F}$. Let $F : \text{cl}(\Omega) \to X$ be a continuous function and assume $F(x, t) \neq x$ for $x \in \text{cl}(\Omega) \sim \Omega$. Assume that F is a k-set-contraction, $k < 1$, in the following sense: for all bounded $A \subset X$, $\gamma(F(\Omega \cap (A \times I))) \leq k\gamma(A)$. Then $i_X(F_t, \Omega_t)$ is defined and $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

PROOF. - By Theorem 2 it suffices to show $S = \{(x, t) \in \Omega : F(x, t) = x\}$ is compact. Since $F(x, t) \neq x$ for $x \in \text{cl}(\Omega) - \Omega$, $S = \{(x, t) \in \text{cl}(\Omega) : F(x, t) = x\}$ and S is closed. Let $T = \{x : (x, t) \in S$ for some $t \in I\}$. Clearly $(T \times I) \cap \Omega \supset S$ so that $F((T \times I) \cap \Omega) \supset F(S) = T$. Thus we have $\gamma(T) \leq \gamma(F((T \times I) \cap \Omega)) \leq k\gamma(T)$, so $\gamma(T) = 0$ and T has compact closure. It follows that $\text{cl}(T) \times I$ is compact, and since $S \subset \text{cl}(T) \times I$, S is compact. Q.E.D.

In order actually to apply Theorem 2 we need some simple conditions under which a homotopy is permissible. We start with a general proposition.

PROPOSITION 1. - Let (M, ρ) be a compact metric space and let A be a closed, bounded subset of a metric space (X, d). Let $F : A \times M \to X$ be a continuous function. For any $m_0 \in M$ assume that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\rho(m, m_0) < \delta$ implies $d(F_m(x), F_{m_0}(x)) < \varepsilon$ for all $x \in A (F_m(x) \equiv F(x, m))$. Further assume that for all $m \in M$, F_m is a k-set-contraction. $k < 1$, k independent of m. Then if $B \subset A$, $\gamma(F(B \times M)) \leq k\gamma(B)$. Furthermore, $\{x \in A : F(x, m) = x$ for some $m \in M\} \equiv S$ is compact or empty.

$= h(y)$, $y \in \partial G$, and $hy - Cy = 0$, a contradiction. Thus the hypotheses of Theorem 3 are satisfied, and we have $i_{X_2}(Ch^{-1}, hG) = i_{X_1}(h^{-1}C, C^{-1}(hG))$.

E. - The fixed point index for local strict-set-contractions.

In this section we shall show that our previous results can be applied in a neat way to give a fixed point index for local strict-set-contractions. We begin with a lemma.

LEMMA 1. - Let G be an open subset of a space $X \in \mathfrak{F}$ and $f: G \to X$ a local strict-set-contraction such that $S = \{x \in G \mid f(x) = x\}$ is compact. Then there exists an open neighborhood V of S such that $K_\infty(f, V)$ is compact. If G is bounded and $f: \mathrm{cl}(G) \to X$ is a k-set-contraction, $k < 1$, such that $g(x) \neq x$ for $x \in \partial G$, then S is necessarily compact.

PROOF. - If f is a local strict-set-contraction and we assume S compact, then for each $x \in S$, there is a bounded open neighborhood N_x such that $f \mid N_x$ is a k_x-set-contraction, $k_x < 1$. Since S is compact there exists a finite open covering of S, say $N_{x_1}, N_{x_2}, \ldots, N_{x_n}$. For convenience set $k_{x_i} = k_i$ and $N_{x_i} = N_i$, $1 \leq i \leq n$. Let $V = \bigcup_{i=1}^{n} N_i$. To show that $K_\infty(f, V)$ is compact, it suffices to show that $f \mid V$ is a k-set-contraction for some $k < 1$. Let $k = \max \{k_i : 1 \leq i \leq n\}$ and let A be any subset of V. We have $f(A) = \bigcup_{i=1}^{n} f(A \cap N_i)$, so that $\gamma(f(A)) = \max \{\gamma(f(A \cap N_i))\} \leq \max \{k\gamma(A \cap N_i)\} \leq k\gamma(A)$.

If G is bounded and $f: \mathrm{cl}(G) \to X$ is k-set-contraction, $k < 1$, such that $f(x) \neq x$ for $x \in \partial G$, $S = \{x \in \mathrm{cl}(G) : f(x) = x\}$ so S is closed. Furthermore, $S = f(S)$, so $\gamma(S) \leq k\gamma(S)$, $k < 1$, and we must have $\gamma(S) = 0$. It follows that S is compact. Q.E.D.

Let G be an open subset of a space $X \in \mathfrak{F}$ and assume $f: G \to X$ is a local strict-set-contraction such that $S = \{x \in G \mid f(x) = x\}$ is compact. By Lemma 1, there exists an open neighborhood V of S such that $K_\infty(f, V)$ is compact. By the results of the previous section there is defined a generalized fixed point index $i_X(f, G) \equiv i_{X \cap K_\infty(f, V)}(f, V \cap X \cap K_\infty(f, V))$. We now examine how Theorems 1-3 of Section D translate to our context.

THEOREM 1. - (The additive property). Let G be an open subset of a space $X \in \mathfrak{F}$ and $f: G \to X$ a local strict-set-contraction such that $S = \{x \in G : f(x) = x\}$ is compact. Assume that $S \subset G_1 \cup G_2$, where G_1 and G_2 are disjoint open subsets of G. Then $i_X(f, G) = i_X(f, G_1) + i_X(f, G_2)$.

PROOF. - By Lemma 2 there exists an open neighborhood V of S such that $\mathrm{cl}(V) \subset G$ and $K_\infty(f, V)$ is compact. By our definition $i_X(f, G) = i_X(f, V)$ and $i_X(f, G_j) = i_X(f, G_j \cap V)$, $j = 1, 2$. By Theorem 1, Section D, $i_X(f, V) =$ $= i_X(f, G_1 \cap V) + i_X(f, G_2 \cap V)$. Q.E.D.

We have also shown that our generalized fixed point index agrees with the classical fixed point index for compact metric ANR's, at least when both are defined. It is natural to ask if our fixed point index agrees with the the LERAY-SCHAUDER fixed point index, or equivalently with the appropriate LERAY-SCHAUDER degree. Let G be an open subset of a BANACH space X and $g: \mathrm{cl}(G) \to X$ a continuous map such that $g(x) \neq x$ for $x \in \partial G$. Assume that g is compact, i.e., $g(G)$ has compact closure. LERAY and SCHAUDER then defined a fixed point index for g (and consequently a degree for $I-g$, $I =$ the identity function). We shall denote this degree by $\deg_{LS}(I-g, G, 0)$. On the other hand, $K_\infty(g, G)$ is certainly compact, so $i_X(g, G)$ is defined.

PROPOSITION 1. $i_X(g, G) = \deg_{LS}(I-g, G, 0)$.

PROOF. - $\deg_{LS}(I-g, G, 0)$ can be defined to be $i_A(g, G \cap A)$, where A is any compact, metric ANR in X containing $g(G)$. (Other definitions of the LERAY-SCHAUDER degree are possible. We shall not prove the equivalence of these definitions here, though it is not hard to do so). In particular, we can take $A = \overline{\mathrm{co}}\, g(G)$. But then $A \supset K_\infty(g, G)$, A is compact and convex, and $g: G \cap A \to A$. Thus we are in the situation of Lemma 1, and $i_A(g, G \cap A) = i_{K_\infty(g, G)}(g, G \cap K_\infty(g, G)) = i_X(g, G)$. Q.E.D.

Let us apply the above result to obtain a proposition of some independent interest. Proposition 2 is used without proof in the BROWDER-NUSSBUAM article [10] in order to identify two different ways of defining a generalized degree. Needless to say, it can be proved directly without the elaborate apparatus assembled here.

PROPOSITION 2. - Let G be an open subset of a BANACH space X_1. Let $C: \mathrm{cl}(G) \to X_2$ be a compact map into a BANACH space X_2. Let $h: \mathrm{cl}(G) \to X_2$ be a homeomorphism such that $h(\mathrm{cl}(G))$ is closed and $h(G)$ is open. Assume that $h(x) - C(x) \neq 0$ for $x \in \mathrm{cl}(G) - G$. Then we have $\deg_{LS}(I - Ch^{-1}, hG, 0)$ and $\deg_{LS}(I - h^{-1}C, C^{-1}(hG), 0)$ are defined and equal.

PROOF. - Let $G_1 = G$ and $G_2 = hG$ and recall that we are assuming G_2 to be open. By Proposition 1 it suffices to show that $i_{X_2}(Ch^{-1}, G_2)$ and $i_{X_1}(h^{-1}C, C^{-1}(G_2))$ are defined and equal. However, notice that we are in the situation of the commutativity property (Theorem 3): $h^{-1}: G_2 \to G_1 \subset X_1$ and $C: G_1 \to X_2$. Thus it suffices to show that the hypotheses of Theorem 3 are met. Since C is compact, it is clear that K_{odd} and K_{even} are compact. We have to show that $S_1 = \{x \in (h^{-1})^{-1}(G_1) \mid (Ch^{-1})(x) = x\} = \{x \in G_2 \mid (Ch^{-1})(x) = x\}$ is compact. However, $h(\mathrm{cl}G)$ is a closed set by assumption, and Ch^{-1} is a compact map defined on $h(\mathrm{cl}G)$, so $\tilde{S}_1 = \{x \in h(\mathrm{cl}G) \mid (Ch^{-1})(x) = x\}$ is compact. But $S_1 = \tilde{S}_1$, for if $(Ch^{-1})(x) = x$ and $x \notin S_1$, then $x \in h(\mathrm{cl}G) - h(G) = h(\partial G)$, $x =$

PROOF. - To prove the first assertion, it suffices to show that $K_n(g_1g_2, H_2) \subset K_{2n-1}$ and $K_n(g_2g_1, H_1) \subset K_{2n}$. The proof is by induction. First consider the case $n = 1$. We have $K_1(g_1g_2, H_2) = \overline{\text{co}}\, g_1(g_2(H_2)) \subset \overline{\text{co}}\, g_1(G_1) = K_1$. Similarly, using that $g_1(H_1) \subset G_2$, we obtain that $K_1(g_2g_1, H_1) \subset \overline{\text{co}}\, g_2(\overline{\text{co}}\, g_1(H_1) \cap G_2) \subset \overline{\text{co}}\, g_2(\overline{\text{co}}\, g_1(G_1) \cap G_2) = K_2$. Generally, suppose we have shown that $K_n(g_1g_2, H_2) \subset K_{2n-1}$ and $K_n(g_2g_1, H_1) \subset K_{2n}$. Then we have that $K_{n+1}(g_1g_2, H_2) = \overline{\text{co}}(g_1g_2)(H_2 \cap K_n(g_1g_2, H_2)) \subset \overline{\text{co}}(g_1g_2)(H_2 \cap K_{2n-1}) = \overline{\text{co}}\, g_1(g_2(H_2 \cap K_{2n-1}) \cap G_1) \subset \overline{\text{co}}\, g_1(\overline{\text{co}}\, g_2(G_2 \cap K_{2n-1}) \cap G_1) = \overline{\text{co}}\, g_1(K_{2n} \cap G_1) = K_{2n+1}$. Similarly, we find that $K_{n+1}(g_2g_1, H_1) = \overline{\text{co}}\,(g_2g_1)(H_1 \cap K_n(g_2g_1, H_1)) \subset \overline{\text{co}}\,(g_2g_1)(H_1 \cap K_{2n}) = \overline{\text{co}}\, g_2(g_1(H_1 \cap K_{2n}) \cap G_2) \subset \overline{\text{co}}\, g_2(\overline{\text{co}}\, g_1(G_2)) = K_{2n+2}$. This completes the inductive step and proves our first assertion.

We have shown above that $K_{2n-1} \supset K_{2n+1}$ and $K_{2n} \supset K_{2n+2}$. To prove our final claim it thus suffices to show that $g_1 : G_1 \cap K_{2n} \to K_{2n+1}$ and $g_2 : G_2 \cap K_{2n-1} \to K_{2n}$. However we have that $g_1(G_1 \cap K_{2n}) \subset \overline{\text{co}}\, g_1(G_1 \cap K_{2n}) = K_{2n+1}$ and $g_2(G_2 \cap K_{2n-1}) \subset \overline{\text{co}}\, g_2(G_2 \cap K_{2n-1}) = K_{2n}$, so we are done. Q.E.D.

THEOREM 3. - Let G_1 and G_2 be open sets in spaces X_1 and X_2 respectively, $X_i \in \mathcal{F}$, $i = 1, 2$. Let $g_1 : G_1 \to X_2$ and $g_2 : G_2 \to X_1$ be continuous maps. Assume $S = \{x \in g_1^{-1}(G_2) \mid (g_2g_1)(x) = x\}$ is compact. Finally, assume that K_{odd} and K_{even} are compact, where the notation is the same as in Lemma 2. Then we have that $i_{X_1}(g_2g_1, g_1^{-1}(G_2))$ and $i_{X_2}(g_1g_2, g_1^{-1}(G_1))$ are defined and equal.

PROOF. - Let $S = \{x \in g_2^{-1}(G_1) \mid (g_1g_2)(x) = x\}$. We know that S_i is compact, $i = 1, 2$, so let H_i be an open neighborhood of S_i such that $\text{cl}(H_1) \subset g_1^{-1}(G_2)$ and $\text{cl}(H_2) \subset g_2^{-1}(G_1)$. By Lemma 2, we know that $K_\infty(g_2g_1, H_1) \subset K_{\text{even}}$ and $K_\infty(g_1g_2, H_2) \subset K_{\text{odd}}$, so that both these sets are compact and $i_{X_1}(g_2g_1, H_1)$ and $i_{X_2}(g_1g_2, H_2)$ are defined.

We also showed in Lemma 2 that $g_2g_1 : H_1 \cap K_{\text{even}} \to K_{\text{even}}$ and $g_1g_2 : H_2 \cap K_{\text{odd}} \to K_{\text{odd}}$. By Lemma 1 of this section, it follows that $i_{X_1}(g_2g_1, H_1) = i_{K^*_{\text{even}}}(g_2g_1, H_1 \cap K^*_{\text{even}})$ and $i_{X_2}(g_1g_2, H_2) = i_{K^*_{\text{odd}}}(g_1g_2, H_2 \cap K^*_{\text{odd}})$, where we have written $K^*_{\text{odd}} \equiv K_{\text{odd}} \cap X_2$ and $K^*_{\text{even}} \equiv K_{\text{even}} \cap X_1$. Thus to complete the proof of Theorem 3 it suffices to show that $i_{K^*_{\text{even}}}(g_2g_1, H_1 \cap K^*_{\text{even}}) = i_{K^*_{\text{odd}}}(g_1g_2, H_2 \cap K^*_{\text{odd}})$. Bu we showed in Lemma 2 that $g_1 : G_1 \cap K_{\text{even}} \to K_{\text{odd}}$ and $g_2 : G_2 \cap K_{\text{odd}} \to K_{\text{even}}$, so it follows that $g_1 : G_1 \cap K^*_{\text{even}} \to K^*_{\text{odd}}$ and $g_2 : G_2 \cap K^*_{\text{odd}} \to K^*_{\text{even}}$. By the commutativity property for the ordinary fixed point index, and the fact that all fixed points of g_2g_1 in $H_1 \cap K^*_{\text{even}} (= S_1)$ lie in $g_1^{-1}(G_2 \cap K^*_{\text{dd}})$ and similarly for g_1g_2, we obtain $i_{K^*_{\text{even}}}(g_2g_1, g_1^{-1}(G_2 \cap K^*_{\text{odd}})) = i_{K^*_{\text{even}}}(g_2g_1, H_1 \cap K^*_{\text{even}}) = i_{K^*_{\text{odd}}}(g_1g_2, g_2^{-1}(g_1 \cap K^*_{\text{even}})) = i_{K^*_{\text{odd}}}(g_1g_2, H_2 \cap K^*_{\text{odd}})$. This completes the proof of our theorem. Q.E.D.

Let us now take stock of our progress. In Theorems 1, 2, and 3 we have proved, respectively, generalizations of the additivity property, the homotopy property, and the commutativity property. We have not generalized the normalization property here, but we shall do so in the next section.

THEOREM 2. - Let $I = [0, 1]$ and let Ω be an open subset of $X \times I$, $X \in \mathfrak{F}$. Let $F : \Omega \to X$ be a continuous map and assume $S = \{(x, t) \in \Omega : F(x, t) = x\}$ is compact. Suppose there exists an open subset 0 of Ω such that $S \subset 0$ and such that if we set $K_1(F, 0) = \overline{\text{co}}\, F(0)$, $K_n(F, 0) = \overline{\text{co}}\, F(0 \cap (K_{n-1} \times I))$, $n > 1$, and $K_\infty(F, 0) = \bigcap_{n \geq 1} K_n(F, 0)$, then $K_\infty(F, 0)$ is compact. Then if we set $\Omega_t = \{x : (x, t) \in \Omega\}$ and $F_t = F(\cdot, t)$, $i_X(F_t, \Omega_t)$ is defined for $t \in I$ and $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

PROOF. - For convenience write $K_\infty = K_\infty(F, 0)$ and $K_\infty(F_t, 0_t) = K_{\infty, t}$, where $0_t = \{x : (x, t) \in 0\}$. It is clear that 0_t is an open neighborhood of the fixed points of F_t and $K_\infty(F_t, 0_t) \subset K_\infty(F, 0)$, a compact set. Thus $i_X(F_t, \Omega_t)$ is defined. As usual, write $K_\infty^* = K_\infty \cap X$ and $K_{\infty, t}^* = K_{\infty, t} \cap X$.

Next observe that $K_n = K_n(F, 0) \supset K_{n+1}$, for $K_1 \supset K_2$ and if $K_n \supset K_{n+1}$, $K_{n+1} = \overline{\text{co}}\, F(0 \cap (K_n \times I)) \supset \overline{\text{co}}\, F(0 \cap (K_{n+1} \times I)) = K_{n+2}$. It is also clear that $F : 0 \cap (K_\infty \times I) \to K_\infty$. For by our construction $F : 0 \cap (K_n \times I) \to K_{n+1}$, so that $F : 0 \cap (K_\infty \times I) \to \bigcap_{n \geq 2} K_n = K_\infty$. Consequently F takes $0 \cap (K_\infty^* \times I)$ to K_∞^*. Considering F as a homotopy on the open set $0 \cap (K_\infty^* \times I)$ in $K_\infty^* \times I$, it is clear that the conditions of Theorem 2, Section C, are met, so that $i_{K_\infty^*}(F_0, 0_0 \cap K_\infty^*) = i_{K_\infty^*}(F_1, 0_1 \cap K_\infty^*)$. However, $K_\infty \supset K_{\infty, t}$, so that by Lemma 1 we have $i_{K_\infty^*}(F_t, 0_t \cap K_\infty^*) = i_{K_{\infty, t}^*}(F_t, 0_t \cap K_{\infty, t}^*) = i_X(F_t, \Omega_t)$. Combining these results, we see that $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

COROLLARY 1. - Let Ω be an open subset of $X \times I$, $X \in \mathfrak{F}$, $I = [0, 1]$. Let $F : \text{cl}(\Omega) \to X$ be a continuous map such that $F(x, t) \neq x$ for $(x, t) \in \text{cl}(\Omega) \sim \Omega$. In the notation of Theorem 2, assume $K_\infty(F, \Omega)$ is compact. Then $i_X(F_0, \Omega_0) = i_X(F_1, \Omega_1)$.

PROOF. - The open set 0 of Theorem 2 is provided by Ω itself. Thus it suffices to show $S = \{(x, t) = x\}$ is compact. Since $F(x, t) \neq x$ for $x \in \text{cl}(\Omega) \sim \Omega$, $S = \{(x, t) = x\}$, and S is closed. Thus we only have to show S is contained in a compact set. However, if $F(x, t) = x$, it is easy to see that $x \in K_\infty(F, \Omega)$, a compact set. Q.E.D.

Before proceding to our next theorem, let us prove some facts about a general construction.

LEMMA 2. - Let G_1 and G_2 be subsets in BANACH spaces B_1 and B_2 respectively. Let $g_1 : G_1 \to B_2$ and $g_2 : G_2 \to B_1$ be continuous maps. Let $K_1 = \overline{\text{co}}\, g_1(G_1)$, $K_2 = \overline{\text{co}}\, g_2(G_2 \cap K_1)$ and generally let $K_{2n+1} = \overline{\text{co}}\, g_1(G_1 \cap K_{2n})$ and $K_{2n+2} = \overline{\text{co}}\, g_2(G_2 \cap K_{2n+1})$. Let $K_{\text{odd}} = \bigcap_{n=1}^\infty K_{2n-1}$ and $K_{\text{even}} = \bigcap_{n=1}^\infty K_{2n}$. Let H_1 be any subset of $g_1^{-1}(G_2)$ and let H_2 be any subset of $g_2^{-1}(G_1)$. Then we have

(1) $K_{\text{even}} \supset K_\infty(g_2 g_1, H_1)$, $K_{\text{odd}} \supset K_\infty(g_1 g_2, H_2)$.

(2) $g_2 g_1 : H_1 \cap K_{\text{even}} \to K_{\text{even}}$, $g_1 g_2 : H_2 \cap K_{\text{odd}} \to K_{\text{odd}}$.

(3) $g_1 : G_1 \cap K_{\text{even}} \to K_{\text{odd}}$, $g_2 : G_2 \cap K_{\text{odd}} \to K_{\text{even}}$.

is an open subset of K_∞^*, and K^* and K_∞^* are compact, metric ANR's, since they are finite unions of compact, convex sets).

PROOF. - By our comment above, $K^* = \bigcup_{i \in F}(K \cap C_i)$, where F is a finite subset of I, $X = \bigcup_{i \in I} C$, and notation is as above. Then we have $K_\infty^* = \bigcup_{i \in F}(K_\infty \cap C_i)$. It is clear that $K_\infty = K_\infty(g, G) \supset K_\infty(g, G \cap K^*)$, so that Theorem 5, Section C, implies $i_{K^*}(g, G \cap K^*) = i_{K_\infty^*}(g, G \cap K_\infty^*)$. Q.E.D.

Now let G be an open subset of a space $X \in \mathfrak{F}$ and $g : \mathrm{cl}(G) \to X$ a continuous function. Assume $g(x) \neq x$ for $x \in \mathrm{cl}(G) - G$ and suppose that $K_\infty(g, G)$ is compact. Let us write $K_\infty^* = X \cap K_\infty(g, G)$ and define the generalized fixed point index of g, $i_X(g, G) = i_{K_\infty^*}(g, G \cap K_\infty^*)$. Notice that K_∞^* is a finite union of compact, convex sets, hence a compact, metric ANR. Also observe that $g(x) \neq x$ for $x \in \mathrm{cl}(G \cap K_\infty^*) \sim G \cap K_\infty^*$, since this set is contained in $(\mathrm{cl}(G) \sim G) \cap K_\infty^*$. Thus we see that $i_{K_\infty^*}(g, G \cap K_\infty^*)$ makes sense.

There is still one problem with our notation, however. If X is a finite union of compact, convex sets, the fixed point index $i_X(g, G)$ is already defined. We must show that $i_X(g, G) = i_{K_\infty^*}(g, G \cap K_\infty^*)$ in that case. Let $K = \overline{\mathrm{co}}\, X$, however, so $K^* = K \cap X = X$ and apply Lemma 1; we obtain $i_X(g, G) = i_{K^*}(g, G \cap K^*) = i_{K_\infty^*}(g, G \cap K_\infty^*)$. Thus our notation is permissible.

We want to show that the generalized fixed point index satisfies properties like those of the ordinary fixed point index.

THEOREM 1. - Let G be an open open subset of a space $X \in \mathfrak{F}$ and g a continuous function, $g : \mathrm{cl}(G) \to X$. Assume that $g(x) \neq x$ for $x \in \mathrm{cl}(G) - G$ and that $K_\infty(g, G)$ is compact. Let $S = \{x \in G \mid g(x) = x\}$ and assume that $S \subset G_1 \cup G_2$ where G_1 and G_2 are two disjoint open sets included in G. Then $i_X(g, G_i)$ is defined, $i = 1, 2$ and $i_X(g, G_1) + i_X(g, G_2) = i_X(g, G)$.

PROOF. - Notice that $K_\infty(g, G_i) \subset K_\infty(g, G)$ so $K_\infty(g, G_i)$ is compact and $i_X(g, G_i)$ is defined. For notational convenience let us write $K_\infty^* = X \cap K_\infty(g, G)$ and $K_{\infty, i}^* = X \cap K_\infty(g, G_i)$. By the ordinary additivity property, we have $i_X(g, G) = i_{K_\infty^*}(g, G \cap K_\infty^*) = i_{K_\infty^*}(g, G_1 \cap K_\infty^*) + i_{K_\infty^*}(g, G_2 \cap K_\infty^*)$. However, by Lemma 1 (where we take the set K to be $K_\infty(g, G_j)$), we obtain $i_{K_\infty^*}(g, G_j \cap K_\infty^*) = i_{K_{\infty, j}^*}(g, G_j \cap K_{\infty, j}^*) = i_X(g, G_j)$. Thus we are done. Q.E.D.

Now suppose $g : G \to X$ is a continuous function, where G is an open subset of a space $X \in \mathfrak{F}$. Notice that we are not assuming g is defined on $\mathrm{cl}(G)$. Assume that $S = \{x \in G \mid g(x) = x\}$ is compact and that there is an open neighborhood V of S such that $\mathrm{cl}(V) \subset G$ and $K_\infty(g, V)$ is compact. Then we define $i_X(g, G) = i_X(g, G)$. Just as in Section B, by using the additivity property and the fact that $i_X(g, G) = 0$ if g has no fixed points in G (this follows immediately from our definition), we see that this definition does not depend on the particular V chosen and agrees with our previous definition when g in defined on $\mathrm{cl}(G)$ and $K_\infty(g, G)$ is compact.

the functions we study in this section may provide greater flexibility in applications. Thus one can prove that the function f considered in Corollary 3 of Section A is a k-set contraction, $k < 1$, with respect to some equivalent norm on the BANACH space, but the proof of this fact is much more difficult than the proof of Corollary 3 itself. However, the proof of Corollary 3 shows simply that f lies in the class we consider, so the results of this section apply to it without proving it is a k-set-contraction, $k < 1$, with respect to an equivalent norm.

Let us begin with a general construction. Let A be a subset of a BANACH space B and let $g: A \to B$ be a continuous map. Let $K_1 = \overline{\mathrm{co}}\, g(A)$, where $\overline{\mathrm{co}}$ denotes the convex closure of a set. Let $K_n = \overline{\mathrm{co}}\, g(A \cap K_{n-1})$, $n > 1$, and let $K_\infty = \bigcap_{n \geq 1} K_n$. We first claim that $K_n \supset K_{n+1}$. This follows because $K_1 = \overline{\mathrm{co}}\, g(A) \supset K_2 = \overline{\mathrm{co}}\, g(A \cap K_1)$ and if $K_{n-1} \supset K_n$, $K_n = \overline{\mathrm{co}}\, g(A \cap K_{n-1}) \supset \overline{\mathrm{co}}\, g(A \cap K_n) = K_{n+1}$. Next notice that $g: A \cap K_\infty \to K_\infty$, for we have $g(A \cap K_n) \subset \overline{\mathrm{co}}\, g(A \cap K_n) = K_{n+1}$ and $K_n \supset K_{n+1}$. Finally, of course, we see that K_∞ is a closed, convex set.

The above construction will be used repeatedly, so we adopt some notation concerning it. We write $K_n = K_n(g, A)$ and $K_\infty = K_\infty(g, A)$. If the set A and the function g are obvious, we may simply write K_n and K_∞ in the sequel.

Before proceeding further let us define the class of metric ANR's X we shall study. Let X be a closed subset of a BANACH space B and assume the norm on B induces the metric on X. Suppose there exists a locally finite cover $\{C_i \mid i \in I\}$ of X by closed, convex sets $C_i \subset X$. More explicitly, suppose we have closed, convex sets $C_i \subset B$, $i \in I$, such that $X = \bigcup_{i \in I} C_i$, and such that for each $x \in X$ there is an open neighborhood 0_x of x such that $0_x \cap C_i$ is empty except for finitely many $i \in I$. If X is as above, we shall write $X \in \mathfrak{F}$. Theorems of HANNER and PALAIS [18, 34] imply that if $X \in \mathfrak{F}$, X is a metric ANR. Notice that if $X \in \mathfrak{F}$ is a metric ANR contained in a BANACH space B and if $K \subset B$ is a compact, convex set, then $X \cap K$ is a finite union of compact, convex sets. For suppose $X = \bigcup_{i \in I} C_i$, C_i a closed, convex set in B. For each $x \in K$, we can find an open neighborhood 0_x such that $0_x \cap C_i = \emptyset$ except for finitely many $i \in I$. Since K is compact, we can cover K by a finite number of these open sets, $0_{x_1}, \ldots, 0_{x_n}$. Then we have $K \cap C_i = \emptyset$ unless $i \in F$, $F = \{i \in I \mid C_i \cap 0_{x_j} \neq \emptyset$ for some j, $1 \leq j \leq n\}$, a finite subset of I, and $X \cap K = \bigcup_{i \in F}(K \cap C_i)$.

Having established notation, let us derive a simple consequence of Theorem 5, Section C.

LEMMA 1. - Let G be an open subset of a metric ANR $X \in \mathfrak{F}$, $X \subset B$, B a BANACH space: Let $g: \mathrm{cl}(G) \to X$ be a continuous map without fixed points on $\mathrm{cl}(g) - G$ and assume that $K_\infty(g, G)$ is compact. Let K be any compact, convex set in B such that $K \supset K_\infty$ and $g: G \cap K \to K$. Let us write $K^*_\infty = K_\infty \cap X$ and $K^* = K \cap X$. Then we have $i_{K^*}(g, G \cap K^*) = i_{K^*_\infty}(g, G \cap K^*_\infty)$. (Notice that this makes sense since $G \cap K$ is an open subset of K^*, $G \cap K^*_\infty$

$f: G \cap A_n \to A_{n+1}$ for $1 \leq n < \infty$ and let $A_\infty = \bigcap_{n\geq 1} A_n$. Then $i_{A_1}(f, G) = i_{A_\infty}(f, G \cap A_\infty)$.

PROOF. - Let G_1 be an open neighborhood of G such that $\mathrm{cl}(G_1) \subset G$. Then $i_{A_1}(f, G) = i_{A_1}(f, G_1)$ and $i_{A_\infty}(f, G \cap A_\infty) = i_{A_\infty}(f, G_1 \cap A_\infty)$. Lemmas 5 and 6, however, imply $i_{A_1}(f, G_1) = i_{A_\infty}(f, G_1 \cap A_\infty)$. Q.E.D.

Using Theorem 4 we can obtain a result which will prove more useful to us.

THEOREM 5. - Let $A = \bigcup_{i=1}^m C_i$ be a finite union of oompact, convex sets C_i in a B-space. Let $B = \bigcup_{i=1}^m D_i$ be a union of compact, convex sets D_i such that $C_i \supset D_i$. Let 0 be an open subset of A and $f: \mathrm{cl}(0) \to A$ a continuous map which has no fixed points on $\mathrm{cl}(0) - 0$. Assume that $f: 0 \cap B \to B$. Finally, let $K_1 = \overline{\mathrm{co}}\, f(0)$, $K_n = \overline{\mathrm{co}}\, f(0 \cap K_{n-1})$, $n > 1$, and $K_\infty = \bigcap_{n\geq 1} K_n$. Assume that $B \supset K_\infty \cap A$. Then we have $i_A(f, 0) = i_B(f, 0 \cap B)$.

PROOF. - Let us set $K_n' = K_n \cap A$. We want to apply Property 5 to the K_n' in order to show that $i_A(f, 0) = i_{K_\infty'}(f, 0 \cap K_\infty')$, where $K_\infty' = \bigcap_{n\geq 1} K_n' = K_\infty \cap A$. First notice that $f: 0 \cap K_n' \to K_{n+1}'$, $n > 0$, for we have $f(0 \cap K_0') = f(0) \subset K_1 \cap A = K_1'$ and, generally, $f(0 \cap K_n') = f(0 \cap K_n) \subset \overline{\mathrm{co}}\, f(0 \cap K_n) \cap A = K_{n+1}'$. Next observe that $K_n \supset K_{n+1}$. The proof is by induction: $K_1 = \overline{\mathrm{co}}\, f(0) \supset \overline{\mathrm{co}}\, f(0 \cap K_1) = K_2$, and if $K_n \supset K_{n+1}$, $K_{n+1} = \overline{\mathrm{co}}\, f(0 \cap K_n) \supset \overline{\mathrm{co}}\, f(0 \cap K_{n+1}) = K_{n+2}$. Thus we have that $K_n' = \bigcup_{i=1}^m (K_n \cap C_i)$ and $(K_n \cap C_i) \supset (K_{n+1} \cap C_i)$. By Property 5, $i_A(f, 0) = i_{K_\infty'}(f, 0 \cap K_\infty') = i_{K_\infty \cap A}(f, 0 \cap K_\infty)$.

Now suppose B is as above. Consider the decreasing suquence of ANR's, $B_0 = B$, $B_n = B \cap K_n$, $n \geq 1$. Since $B \supset K_\infty \cap A$, it is easy to see that $\bigcap_{n\geq 1} B_n = K_\infty \cap A = K_\infty'$. Since $f: 0 \cap K_n \to K_{n+1}$, $f: 0 \cap B_n \to B_{n+1}$. Finally, $B_n = \bigcup_{i=1}^m (D_i \cap K_n)$, and $D_i \cap K_n \supset D_i \cap K_{n+1}$. By Property 5, $i_B(f, 0 \cap B) = i_{K_\infty'}(f, 0 \cap K_\infty')$. Q.E.D.

We now have those refinements of the classical fixed point index which we shall need, and we can proceed to more significant generalizations.

D. - The fixed point index for functions like k-set-contractions.

In this section we shall define a fixed point index for certains functions which behave like k-set-contractions, $k < 1$. Our primary goal is to lay the groundwork for defining the fixed point index for k-set-contractions, $k < 1$, and then for « nice » 1-set-contractions (including condensing maps). However, some of our results here may have independent interest. It will not be hard to see that the class of functions we shall consider includes functions which may not be k-set-contractions, $k < 1$, with respect to any equivalent norm; and in fact such maps actually arise in applications. (See [30]). Furthermore, REMARK. - The strength of the above assumption is that not only is A_∞ a deformation retract of A_n for n large enough, but a deformation retraction H_n can be chosen which moves points very little, i.e., $\|H_n(x, t) - x\| < \delta$ for $x \in A_n$, $t \in I$. If we assumed f defined on all of A_1 and $f: A_n \to A_{n+1}$ (so that the various fixed point indices would be LEFSCHETZ numbers), then we would only need A_∞ a deformation retract of A_n for n large enough in order to show $i_{A_1}(f, A_1) = \Lambda_{A_1}(f) = \Lambda_{A_\infty}(f) = i_{A_\infty}(f, A_\infty)$.

PROOF. - $f(x) \neq x$ for $x \in \mathrm{cl}(G) - G$, and $\mathrm{cl}(G) - G$ is compact, so we can find $\delta > 0$ such that $\|f(x) - x\| \geq \delta > 0$ for $x \in \mathrm{cl}(G) - G$. Select H_n as above for this δ, so that H_n is defined for $n \geq n(\delta)$ and $\|H_n(x, t) - x\| < \delta$ for $x \in A_n$, $t \in I$.

By using Property 4′ repeatedly, we see that $i_{A_1}(f, G) = i_{A_2}(f, G \cap A_2) = i_{A_{n(\delta)}}(f, G \cap A_{n(\delta)})$. (Notice that if $A_n \cap G$ is empty for some n, f has no fixed points in G, since all fixed points lie in $A_\infty \cap G \subset A_n \cap G$. Thus in the case that $A_n \cap G$ is empty for some n, $i_{A_m}(f, A_m \cap G) = 0$ for all m, and the lemma is proved). Consider the homotopy $F(x, t) = H_{n(\delta)}(f(x), t)$, $F: A_{n(\delta)} \times I \to A_{n(\delta)}$. We have $\|H_{n(\delta)}(f(x), t) - f(x)\| < \delta$ for $x \in \mathrm{cl}(G \cap A_{n(\delta)}) - G \cap A_{n(\delta)}$. It follows that $F(x, t) \neq x$ for $x \in \mathrm{cl}(G \cap A_{n(\delta)}) - G \cap A_{n(\delta)}$, so $i_{A_{n(\delta)}}(F(\cdot, 0), G \cap A_{n(\delta)}) = i_{A_{n(\delta)}}(f, G \cap A_{n(\delta)}) = i_{A_{n(\delta)}}(F(\cdot, 1), G \cap A_{n(\delta)})$. However, $F(x, 1) \in A_\infty$ for $x \in G \cap A_{n(\delta)}$, so by Property 4′, $i_{A_{n(\delta)}}(F(\cdot, 1), G \cap A_{n(\delta)}) = i_{A_\infty}(F(\cdot, 1) | G \cap A_\infty, G \cap A_\infty) = i_{A_\infty}(f, G \cap A_\infty)$. Q.E.D.

Of course the reason for proving Lemma 5 is that its hypotheses can be verified in a case which will be of interest to us.

LEMMA 6. - Let $A_n = \bigcup_{i=1}^m C_{i,n}$ be a union of m closed, convex sets in some fixed BANACH space X, *m independent of n*. Assume that A_n is bounded, $\lim \gamma(A_n) = 0$ and $C_{i,n} \supset C_{i,n+1}$ for $1 \leq i \leq m$. (We allow the possibility that $C_{i,n}$ is empty). Then for any $\delta > 0$, there exists a sequence of deformation retractions $H_n: A_n \times I \to A_n$ defined for $n \geq n(\delta)$ and meeting the conditions of Lemma 5.

PROOF. - By Corollary 2 of Section B, given $\delta > 0$, there there exists $n(\delta)$ such that for $n \geq n(\delta)$ there exists a retraction $R_n: A_n \to A_\infty = \bigcap_{n \geq 1} A_n$ such that $\|R_n(x) - x\| < \delta$ for $x \in A_n$ and $R_n(x) \in C_{i,n}$ if $x \in C_{i,n}$. Set $H_n(x, t) = (1 - t)x + tR_n(x)$. It is clear that H_n satisfies the required conditions.

Q.E.D.

THEOREM 4. - Let $A_n = \bigcup_{i=1}^m C_{i,n}$ be a union of m compact, convex sets $C_{i,n}$ in a fixed BANACH space X, m independent of n, $1 \leq n < \infty$. Assume that $C_{i,n} \supset C_{i,n+1}$ for $1 \leq n < \infty$. Let G be an open subset of A_1, $f: G \to A_1$ a continuous map such that $S = \{x \in G : f(x) = x\}$ is compact. Assume that

$j_1(f_1^{-1}(0_2)) = i_{D_1}(g_2 g_1, g_1^{-1}(U_2))$ and similarly for A_2 and D_2. This implies that in proving Theorem 3 we can assume A_1 and A_2 are subsets of BANACH spaces X_1 and X_2 respectively.

Since A_i is an ANR, A_i is a retract of some open neighbourhood U_i in X_i by a retraction $r_i : U_i \to A_i$, $i = 1, 2$. Cover A_i by a finite number of open balls B_{ij}, $1 \leq j \leq k_i$, such that $\text{cl}(B_{ij}) \subset U_i$, $i = 1, 2$. Then we see that $\bigcup_{j=1}^{k_i} \overline{\text{co}}(B_{ij} \cap A_i) \equiv B_i$ is a finite union of compact, convex sets, $B_i \supset A_i$, and $r_i : B_i \to A_i$. Henceforth we shall view the maps r_i as defined only on B_i.

We have $f_1 r_1 : r_1^{-1}(0_1) \to A_2 \subset B_2$ and $f_2 r_2 : r_2^{-1}(0_1) \to A_1 \subset B_1$. If $\tilde{S}_1 = \{x \in (f_1 r_1)^{-1}(r_2^{-1}(0_2)) : (f_2 r_2)(f_1 r_2)(x) = x\}$, we easily check that $\tilde{S}_1 = \{x \in r_1^{-1}(f_1^{-1}(0_2)) : (f_2 r_2)(f_1 r_1)(x) = x\} = \{x \in f_1^{-1}(0_2) : (f_2 f_1)(x) = x\} = S_1$, so that $\tilde{S}_1$ is a compact subset of $(f_1 r_1)^{-1}(r_2^{-1}(0_2)) = r_1^{-1}(f_1^{-1}(0_2))$.

Similary, we check that $\tilde{S}_2 = \{x \in (f_2 r_2)^{-1}(r_1^{-1}(0_1)) : (f_1 r_1)(f_2 r_2)(x) = x\}$ is equal to S_2 and is a compact subset of $r_2^{-1}(f_2^{-1}(0_1))$. Applying Lemma 4 we find that $i_{B_1}[(f_2 r_2)(f_1 r_1), (f_1 r_1)^{-1}(r_2^{-1}(0_2))] = i_{B_2}[(f_1 r_1)(f_2 r_2), (f_2 r_2)^{-1}(r_1^{-1}(0_1))]$. However, it is easy to see that $(f_2 r_2)(f_1 r_1) = (f_2 f_1)(r_1)$, $(f_1 r_1)(f_2 r_2) = (f_1 f_2)(r_2)$, $(f_1 r_1)^{-1}(r_2^{-1}(0_2)) = (r_1^{-1}(f_1^{-1}(0_2))$ and $(f_2 r_2)(r_1^{-1}(0_1) = r_2^{-1}(f_2^{-1}(0_1))$. Substituting these results in the above equality, we obtain $i_{B_1}[(f_2 f_1)(r_1), r_1^{-1}(f_1^{-1}(0_2))] = i_{B_2}[(f_1 f_2)(r_2), r_2^{-1}(f_2^{-1}(0_1))]$.

Now we apply the ordinary commutativity property again:

We have $r_1 : B_1 \to A_1$, $(f_2 f_1) : f_1^{-1}(0_2) \to A_1 \subset B_1$, and the other necessary conditions are met, so $i_{A_1}[(r_1)(f_2 f_1), f_1^{-1}(0_2)] = i_{A_1}(f_2 f_1, f_1^{-1}(0_2)) = i_{B_1}[(f_2 f_1)(r_1), r_1^{-1}(f_1^{-1}(0_2))]$. Similarly, we obtain $i_{A_2}(f_1 f_2, f_2^{-1}(0_1)) = i_{B_2}[(f_2 f_1)(r_2), r_2^{-1}(f_2^{-1}(0_1))]$.

Q.E.D.

We have now obtained the desired refinements of the fundamental properties of the fixed point index. We next establish some special properties which will be crucial for our further work.

Let us begin by considering the following general situation: Let $A = A_1$, $A_2, \ldots, A_n, \ldots$ be a decreasing sequence of compact ANR's in a BANACH space (we allow the possibility that the A_n's are empty for $n \geq n_0$). Assume that $A_\infty = \bigcap_{n \geq 1} A_n$ is an ANR. Let G be an open subset of A and $f : \text{cl}(G) \to A$ a continuous map such that $f(x) \neq x$ for $x \in \text{cl}(G) - G$ and $f : G \cap A_n \to A_{n+1}$. By applying Property 4′ repeatedly, it is easy to show that $i_A(f, G) = i_{A_n}(f, G \cap A_n)$. It is thus natural to ask if $i_A(f, g) = i_{A_\infty}(f, G \cap A_\infty)$. We do not know exactly when it is true that $i_A(f, G) = i_{A_\infty}(f, G \cap A_\infty)$. The following sufficient condition will be satisfactory for our purposes, however.

LEMMA 5. - Let A_n, f, and G be as above. Assume that for any $\delta > 0$ there exists a sequence of deformation retractions $H_n : A_n \times I \to A_n$, defined for $n \geq n(\delta)$ and such that $H_n(\cdot, 0)$ is the identity on A_n, $H_n(\cdot, 1)$ is a retraction on A_∞, $H_n(x, t) = x$ for $x \in A_\infty$ and $t \in I$, and $\|H_n(x, t) - x\| < \delta$ for $x \in A_n$, $t \in I$. Then $i_A(f, G) = i_{A_\infty}(f, G \cap A_\infty)$.

the set of fixed points of $f_2 f_1$ in $f_1^{-1}(0_2)$ is a compact subset of $f_1^{-1}(0_2)$. Then $i_{A_1}(f_2 f_1, f_1^{-1}(0_2)) = i_{A_2}(f_1 f_2, f_2^{-1}(0_1))$.

REMARK. - The fact that A_i is an ANR follows from two theorems. First, by a theorem of DUGUNDJI [15], a closed, convex subset of a BANACH space is an AR. Second, if two topological spaces (both contained in some topological space) are ANRs and their intersection is also an ANR, then their union is an ANR.

PROOF. - By Lemma 1, the fixed point set of $f_1 f_2$ in $f_2^{-1}(0_1)$ is a compact subset of $f_2^{-1}(0_1)$, so $i_{A_2}(f_1 f_2, f_2^{-1}(0_1))$ is defined. Let S_2 denote the fixed point sel of $f_1 f_2$ in $f_2^{-1}(0_1)$ and recall that $f_2 . S_2 \to S_1$ is a homeomorphism.

Let U_2 be an open neighborhood of S_2 such that $\mathrm{cl}(U_2) \subset f_2^{-1}(0_1)$. Let U_1 be an open neighborhood of S_1 such that $\mathrm{cl}(U_1) \subset f_1^{-1}(0_2)$ and $\mathrm{cl}[f_1(U_1)] \subset 0_2$. To obtain such a U_1 we simply take an open neighborhood V of S_2 such that $\mathrm{cl}(V) \subset 0_2$, and we put $U_1 = f_1^{-1}(V)$. Since $\mathrm{cl}[f_1(U_1)]$ is a compact subset of 0_2, we can cover it by a finite number of balls in X_2, say $B_1, \dots, B_n$ such that $\mathrm{cl}(B_i \cap A_2) \subset 0_2$, $1 \leq i \leq n$. Since A_2 is a finite union of compact, convex pieces, $\mathrm{cl}(B_i \cap A_2)$ is a finite union of compact, convex pieces, and consequently so is $A = \bigcup_{i=1}^{n} \mathrm{cl}(B_i \cap A_2)$.

With these preliminary constructions, we can complete our proof. By definition, $i_{A_1}(f_2 f_1, f_1^{-1}(0_2)) = i_{A_1}(f_2 f_1, U_1)$. On the other hand, by construction, $f_1(U_1) \subset A$; and since $A \subset 0_2$, f_2 is defined on A and $f_2 : A \to A_1$. Thus we can apply the ordinary commutativity property, and we obtain $i_A[f_1 f_2, (f_2 | A)^{-1}(U_1)] = i_{A_1}(f_2 f_1, U_1)$. We now have to show that $i_A[f_1 f_2, (f_2 | A)^{-1}(U_1)] = i_{A_2}(f_1 f_2, U_2)$. If we can show this, we are done, for by definition $i_{A_2}(f_1 f_2, U_2) = i_{A_2}(f_1 f_2, f_2^{-1}(0_1))$. Thus, suppose that $(f_1 f_2)(x) = x$ for $x \in f_2^{-1}(0_1)$, i.e., $x \in S_2$. Since $f_2 : S_2 \to S_1$ is a homeomorphism, $x = f_2^{-1}(z)$ for some $z \in S_1 \subset U_1$; in other words, we have $S_2 \subset f_2^{-1}(U_1)$. It follows by the additivity property that $i_{A_2}(f_1 f_2, U_2) = i_{A_2}(f_1 f_2, f_2^{-1}(U_1) \cap U_2)$ However, $f_1 f_2 : f_2^{-1}(U_1) \cap U_2 \to A \subset A_2$, so by Property 4', $i_{A_1}(f_1 f_2, f_2^{-1}(U_1) \cap U_2) = i_A(f_1 f_2, f_2^{-1}(U_1) \cap U_2 \cap A)$. We see easily, however, that $f_2^{-1}(U_1) \cap \cap A = (f_2 | A)^{-1}(U_1)$, and since $S_2 \subset U_2 \cap (f_2 | A)^{-1}(U_1)$, the additivity property implies that $i_A(f_1 f_2, (f_2 | A)^{-1}(U_1)) = i_A(f_1 f_2, f_2^{-1}(U_1) \cap U_2 \cap A)$. Q.E.D.

THEOREM 3. - Let A_1 and A_2 be compact metric ANR's, 0_1 and 0_2 open subsets of A_1 and A_2 respectively, and $f_1 : 0_1 \to A_2$ and $f_2 : 0_2 \to A_1$ continuous maps. Assume that $S_1 = \{x \in f_1^{-1}(0_2) : (f_2 f_1)(x) = x\}$ is compact. Then $i_{A_2}(f_1 f_2, f_2^{-1}(0_1))$ is defined and $i_{A_2}(f_1 f_2, f_2^{-1}(0_1)) = i_{A_1}(f_2 f_1, f_1^{-1}(0_2))$.

PROOF. - By Lemma 2 there exist isometric imbeddings j_1 and j_2 of A_1 and A_2 respectively into BANACH spaces X_1 and X_2. If we set $U_i = j_i(0_i)$, $D_i = j_i(A_i)$, $(i = 1, 2)$, $g_1 = j_2^{-1} f_1 j_2^{-1}$ and $g_2 = j_1 f_2 j_2^{-1}$, and if we note that $g_1^{-1}(U_2) = j_1(f_1^{-1}(0_2))$, then Lemma 3 implies that $i_{A_1}(f_2 f_1, f_1^{-1}(0_2)) = i_{D_1}(j_1(f_2 f_1) j_1^{-1},$

and f_1 restricted to S_1 is a homeomorphism of S_1 onto S_2 whose inverse is f_2 restricted to S_2. In particular, S_1 is compact iff S_2 is compact.

PROOF. - If $x \in S_1$, $(f_2f_1)(x) = x$ so $(f_1f_2)(f_1(x)) = f_1(x)$ and $f_1(x) \in S_2$. Conversely, if $y \in S_2$, $(f_1f_2)(y) = y$, so $(f_2f_1)(f_2(y)) = f_2(y)$ and $f_2(y) \in S_1$. Thus we have $S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} S_1$. However, if $x \in S_1$, $(f_2f_1)(x) = x$, and if $y \in S_2$, $(f_2f_1)(y) = y$, so f_1f_2 is the identity on S_1 and f_1f_2 is the identity on S_2 and f_1f_2 is the identity on S_2, i.e., $f_1 | S_1$ is a homeomorphism. Q.E.D.

Our next lemma is a standard result, but we include the proof for completeness.

LEMMA 2. - Any complete metric space (X, d) can be isometrically imbedded as a closed subset of BANACH space B.

PROOF. - Consider the space B of real valued, bounded, continuous functions on X. If $f \in B$, $\|f\| = \sup_{x \in X} |f(x)|$. Take a fixed point $x_0 \in X$, and for $x \in X$, consider the function $f_x \in B$ given by $f_x(y) = d(y, x) - d(y, x_0)$. First, notice that f_x is bounded, for $|f_x(y)| \leq d(x, x_0)$. Clearly f_x is continuous. We now claim that the map $j : x \to f_x$ is an isometry. For, take x_1 and $x_2 \in X$ and consider $\|f_{x_1} - f_{x_2}\|$. For $y \in X$ we have that $|f_{x_1}(y) - f_{x_2}(y)| = |d(y, x_1) - d(y, x_0) - d(y, x_2) + d(y, x_0)| = |d(y, x_1) - d(y, x_2)| \leq d(x_1, x_2)$ so $\|f_{x_1} - f_{x_2}\| \leq d(x_1, x_2)$. Conversely, if we take $y = x_2$ we see that $|f_{x_1}(y) - f_{x_2}(y)| = |d(x_2, x_1) - d(x_2, x_0) - d(x_2, x_2) + d(x_2, x_0)| = d(x_1, x_2)$, so $\|f_{x_1} - f_{x_2}\| \geq d(x_1, x_2)$.

Finally, notice that jX is a closed subset of B. Eor suppose that $f_{x_n} \to g$. Then f_{x_n} is a CAUCHY sequence, so x_n is a CAUCHY sequence. Since X is complete, $x_n \to x$ and consequently $f_{x_n} \to f_x = g$. Q.E.D.

LEMMA 3. - Let A be a compact metric ANR and let $h : A \to A_1$ be a homeomorphism of A onto a topological space A_1. Let G be an open subset of A and $f : G \to A$ a continuous map such that $S = \{x \in G : f(x) = x\}$ is compact. Let $G_1 = h(G)$ and $f_1 = hfh^{-1} : h(G_1) \to A_1$. Then $i_A(f, G) = i_{A_1}(f_1, G_1)$.

PROOF. - Notice that $i_{A_1}(f_1, G_1)$ is defined, since $S_1 = \{y \in G_1 : f_1(y) = y\} = h(S)$. Let V be an open neighborhood of S such that $\mathrm{cl}(V) \subset G$ and let $V_1 = h(V)$. By definition we have $i_A(f, G) = i_A(f, V)$ and $i_{A_1}(f_1, G_1) = i_{A_1}(f_1, V_1)$. To show $i_A(f, V) = i_A(f_1, V_1)$ we use the commutativity property. Since $f_1 = h(fh^{-1}) \equiv hg$ and $h : A \to A_1$ while $g : V_1 \to A$, commutativity implies $i_{A_1}(hg, V_1) = i_A(gh, h^{-1}(V_1)) = i_A(f, V)$. Q.E.D.

LEMMA 4. - Let A_1 and A_2 be finite unions of compact, convex sets in BANACH space X_1 and X_2 respectively. It is known then that A_1 and A_2 are compact metric ANR's. Let 0_1 and 0_2 be open subsets of A_1 and A_2 respectively and let $f_1 : 0_1 \to A_2$ and $f_2 : 0_2 \to A_1$ be continuous maps. Assume that

REMARK. - Notice that if $\Omega = G \times I$ and F is a continuous function from $\mathrm{cl}(G) \times I$ to A such that $F(x, t) \neq x$ for $x \in \mathrm{cl}(G) \sim G$, then $S = \{(x, t) \in G \times I : F(x, t) = x\} = \{(x, t) \in \mathrm{cl}(G) \times I : F(x, t) = x\}$ is compact.

PROOF. - Suppose we can show that every $t \in I$ has an open neighborhood (open in I), 0_t, such that $i_A(F_s, \Omega_s)$ is constant for $s \in 0_t$. If we let $U = \{t \in I : i_A(F_t, \Omega_t) = i_A(F_0, \Omega_0)\}$, it then follows that U is open in I. But U is the complement in I of $V = \{t \in I : i_A(F_t, \Omega_t) \neq i_A(F_0, \Omega_0)\}$, and the same reasoning implies that V is open. Thus U is open and closed and hence $U = I$.

Thus we have only to show that given $t_0 \in I$, there is an open neighborhood 0_{t_0} such that $i_A(F_t, \Omega_t)$ is constant for $t \in 0_{t_0}$. Let $S_t = \{(x, t) | (x, t) \in S\}$. Given $(x, t_0) \in S_{t_0}$, we can find an open neighborhood $N_x \times (J_x \cap I)$ of (x_0, t_0) (open in the topology of Ω) such that $\mathrm{cl}(N_x) \times \mathrm{cl}(J_x \cap I) \subset \Omega$, N_x is an open neighborhood in A of x and $J_x = (t_0 - \varepsilon_x, t_0 + \varepsilon_x)$ is an open interval about t_0. Since S_{t_0} is compact, we can over it by a finite number of these neighborhoods, say $N_{x_i} \times (J_{x_i} \cap I)$, $1 \leq i \leq n$. Let $\varepsilon = \min\{\varepsilon_{x_i}, 1 \leq i \leq n\}$, let $V = \bigcup_{i=1}^{n} N_{x_i}$, and let $J = (t_0 - \varepsilon, t_0 + \varepsilon)$. Then $S_{t_0} \subset V \times (J \cap I)$ and $\mathrm{cl}(V) \times \mathrm{cl}(J \cap I) \subset \Omega$.

Let $J_\eta = (t_0 - \eta, t_0 + \eta)$. We claim that for η small enough, $S_t \subset V \times \mathrm{cl}(J_\eta \cap I)$ for $t \in \mathrm{cl}(J_\eta \cap I)$. The proofs is by contradiction. Suppose not. Then we can find $(x_n, t_n) \in S$ such that $t_n \to t_0$ but $x_n \notin V$. Since S is compact, we can find a convergent subsequence $(x_{n_i}, t_{n_i}) \to (x, t_0)$. By the continuity of F, $F(x, t_0) = x$, so $(x, t_0) \in S_{t_0}$. But $x_{n_i} \notin V$, so $x \notin V$ since V is open. This is a contradiction.

We are almost finished. Select V and η as above. F is a continuous function on $\mathrm{cl}(V) \times \mathrm{cl}(J_\eta \cap I)$, and, of oourse, $\mathrm{cl}(J_\eta \cap I)$ is a closed interval of real numbers. By construction we have $F(x, t) \neq x$ for $x \in \mathrm{cl}(V) - V$ and $t \in \mathrm{cl}(J_\eta \cap I)$. Thus, by the ordinary homotopy property we find $i_A(F_t, V)$ is constant for $t \in \mathrm{cl}(J_\eta \cap I)$. However, for $t \in \mathrm{cl}(J_\eta \cap I)$, V is an open neighborhood of $\{x | F_t(x) = x\} = \{x | (x, t) \in S_t\}$, and we know that $\mathrm{cl}(V) \subset \Omega_t$. Thus we have $i_A(F_t, V) = i_A(F_t, \Omega_t)$. Q.E.D.

The normalization property in our new context is the same as before, and we do not repeat its statement.

Thr proof of the generalized version of the multiplication formula is somewhat more involved than the previous results. First we need a few simple lemmas.

LEMMA 1. - Let G_i be an open subset of a topological space A_i, $i = 1, 2$. Let $f_1 : G_1 \to A_2$ and $f_2 : G_2 \to A_1$ be continuous maps. Let $S_1 = \{x \in f_1^{-1}(G_2) : (f_2 f_1)(x) = x\}$ and let $S_2 = \{x \in f_2^{-1}(G_1) : (f_1 f_2)(x) = x\}$. Then f_1 takes S_1 into S_2

As a special case of Property 4, we have the following property, which we shall use repeatedly:

4'. - Let A and B belong to a, $A \subset B$. Let $f : A \to B$ be the inclusion. Let G be and open subset of B and $g : \mathrm{cl}(G) \to A$ be a continuous map with no fixed points on $\mathrm{cl}(G) - G$. Then we have $i_B(fg, G) = i_B(g, G) = i_A(gf, f^{-1}(G)) = i_A(g, G \cap A)$.

Let us begin our generalizations with a slight widening of the definition of the fixed point index. Let A belong to a and let G be an open subset of A. Let $g : G \to A$ be a continuous map and assume that $S = \{x \in G \mid g(x) = x\}$ is a compact subset of G. (S may be empty. By compact we shall always mean compact or empty). Clearly we can select an open neighborhood V of S such that $\mathrm{cl}(V) \subset G$. We define $i_A(g, G) = i_A(g, V)$. We have to show that this definition does not depend on the particular V chosen and agrees with our previous definition when g is defined on $\mathrm{cl}(G)$. To see that the definition is independent of V, let V_1 and V_2 be two open neighborhoods of S with $\mathrm{cl}(V_i) \subset G$. Then $V_1 \cap V_2$ is an open neighborhood of S and $\mathrm{cl}(V_1 \cap V_2) \subset G$. If we consider the disjoint open sets $V_i \sim \mathrm{cl}(V_1 \cap V_2)$ and $V_1 \cap V_2$, since $S \subset V_1 \cap V_2$ it is clear that the additivity property applies and $i_A(g, V_i) = i_A(g, V_i \sim \mathrm{cl}(V_1 \cap V_2)) + i_A(g, V_1 \cap V_2)$. Since f has no fixed points in $V_i \sim \mathrm{cl}(V_1 \cap V_2)$, $i_A(g, V_i \sim \mathrm{cl}(V_1 \cap V_2)) = 0$ and $i_A(g, V_1) = i_A(g, V_1 \cap V_2) = i_A(g, V_2)$. If g is defined on $\mathrm{cl}(G)$ and V is selected as above, $i_A(g, G) = i_A(g, G - \mathrm{cl}(V)) = i_A(g, V)$. Thus our definition agrees with the usual one when g is defined on $\mathrm{cl}(G)$.

We want to establish generalisations of the four properties listed above.

THEOREM 1. - Let A belong to a, G be an open subset of A, and $f : G \to A$ be a continuous map. Assume that $S = \{x \in G \mid f(x) = x\}$ is a compact subset of G and that $S \subset G_1 \cup G_2$ where G_i are disjoint open subsets of G. Then $i_A(f, G) = i_A(f, G_1) + i_A(f, G_2)$.

PROOF. - Notice that $S_1 = \{x \in G_1 : f(x) = x\}$ is a compact subset of G_1, because $S_1 = \{x : x \in S \cap (A \sim G_2)\}$. Similarly, $S_2 = \{x \in G_2 : f(x) = x\}$ is compact. Thus we see $i_A(f, G_i)$ is defined.

Let V_i be an open neighborhood of S_i such that $\mathrm{cl}(V_i) \subset G_i$. Since G_1 and G_2 are disjoint, V_1 and V_2 are disjoint. Let $V = V_1 \cup V_2$. V is an open neighborhood of S, and $\mathrm{cl}(V) = \mathrm{cl}(V_1) \cup \mathrm{cl}(V_2) \subset G$. By the ordinary additivity property, we have $i_A(f, G_1) + i_A(f, G_2) \equiv i_A(f, V_1) + i_A(f, V_2) = i_A(f, V) \equiv i_A(f, G)$.

THEOREM 2. - Let A belong to a and let $I = [0, 1]$. Let Ω be an open subset of $A \times I$ and $F : \Omega \to A$ a continuous map. Assume that $S = \{(x, t) \in \Omega : F(x, t) = x\}$ is compact. Let $\Omega_t = \{x \in A : (x, t) \in \Omega\}$ and $F_t = (F \cdot, t)$. Then we have $i_A(F_0, \Omega_0) = i_A(F_1, \Omega_1)$.

by Theorem 1 to find an integer n_0 such that for $n \geq n_0$, $C_{J,\infty}$ is empty iff $C_{J,n}$ is empty. However, $C_{J,n}$ is a decreasing sequence of closed sets in a complete metric space Y, $\gamma(C_{J,n}) \to 0$ and $C_{J,\infty} = \bigcap_{n\geq 1} C_{J,n}$. It follows by Kuratowski's theorem that if $C_{J,\infty}$ is empty, $C_{J,n}$ is empty for $n \geq n_J$. Selecting $n_0 \geq \max\{n_J\}$, we are done.

To prove the second part of the Corollary, let $\tilde{O}_K$ be the open neighborhood of $C_{K,\infty}$ guaranteed by Theorem 1. By Theorem 1, it suffices to find an integer $n_0 \geq N_0$ such that $C_{K,n} \subset \tilde{O}_K$ for $n \geq n_1$. However, by Kuratowski's theorem again, there exists an integer n_K such that $C_{K,n} \subset \tilde{O}_K$ for $n \geq n_K$. We merely take $n_1 \geq \max\{n_0, n_K\}$. Q.E.D.

C. - The classical fixed point index.

Our goal in the next few sections is to define a fixed point index for k-set-contraction, $k < 1$, defined on certain «nice» absolute neighborhood retracts (ANR's). Let us begin by recalling the basic properties of the fixed point index [6]. Let a be the category of compact metric absolute neighborhood retracts (ANR's) and continuous mappings. Let A belong to a, G be an open subset of A, and $f : \mathrm{cl}(G) \to A$ be a continuous function which has no fixed points on $\mathrm{cl}(G) - G$. Then there is a unique integer valued function $i_A(f, G)$ which satisfies the following four properties:

1. - If $f : \mathrm{cl}(G) \to A$ has no fixed point on $\mathrm{cl}(g) - G$, and the fixed point of f in G lie in $G_1 \cup G_2$, where G_1 and G_2 are two disjoint open sets included in G, then $i_A(f, G) = i_A(f, G_1) + i_A(f, G_2)$. In particular, if f has no fixed points in G, this is meant to say that $i_A(f, G) = 0$. (The additivity property).

2. - Let I denote the clooed unit interval $[0, 1]$. If $F : \mathrm{cl}(G) \times I \to A$ (A belongs to a, of course) is a continuous map, and $F_t(x) = F(x, t)$ has no fixed points on $\mathrm{cl}(G) - G$ for $0 \leq t \leq 1$, then $i_A(F_0, G) = i_A(F_1, G)$. (The homotopy property).

3. - If $G = A$, then $i_A(f, G) = \Lambda(f)$. $\Lambda(f)$, the LEFSCHETZ number of f, equals $\sum(-1)^k$ trace (f_{*k}), where $f_{*k} : H_k(A) \to H_k(A)$ is the vector space homomorphism of $H_k(A)$ to $H(A)$ and $H_k(A)$ is the Cech homology of A with rational coefficients. (The normalization property).

4. - Let A and B be two spaces which belong to a. Let $f : A \to B$ be a continuous map. Let V be an open subset of B and $g : \mathrm{cl}(V) \to A$ a continuous map. Assume fg has no fixed points on $\mathrm{cl}(V) - V$. Let $U = f^{-1}(V)$. Then gf has no fixed points on $\mathrm{cl}(U) - U$ and $i_B(fg, V) = i_A(gf, U)$. (The commutativity property).

Next let us show that R is actually a retraction of C onto D. Since $R(x) = s(x)$ for $x \in W_1$, to show R is a retraction it suffices to show $R(x) \in D$ for $x \in C$. Given $x \in C$, select J such that $x \in U_J$ but $x \notin U_L$ for $|L| > |J|$. Furthermore, if $|L| \geq |J|$, we know that $U_L \cap U_J$ is empty unless $L \supset J$, so that $\theta_L(x) = 0$ unless $L \supset J$. Now consider $R(x) = \theta(x)s(x) + \Sigma_K \theta_K(x) r_K(x)$. If $\theta(x) = 1$, $R(x) = s(x) \in D$. If $\theta(x) < 1$, $R(x) = \theta(x)s(x) + (1-\theta))\sum_K(\theta_K(x)/1 - \theta(x)r_K(x)$. Since $r_K(x) \in D_K \subset D_J$ for $K \supset J$ and since $\theta_K(x) = 0$ unless $K \supset J$, $\sum_K(\theta_K(x)/1 - \theta(x))r_K(x)$ is a convex combination of points in D_J and hence lies in D_J. If $\theta(x) = 0$, we are done. If $\theta(x) > 0$, we have $x \in W$, so that $s(x) \notin C_k$ for $k \notin J$ and consequently $s(x) \notin D_k$ for $k \notin J$. It follows that $s(x) \in D_j$ for some $j \in J$. Thus $R(x)$ is a convex combination of points in D_j and lies in $D_j \subset D$.

Finally, let us suppose that $C_K \subset \tilde{O}_K$ for all K. Then by our construction $\theta_K(x) = 0$ unless $p(r_K(x) - x) < \varepsilon$. Also by construction, $\theta(x) = 0$ unless $p(s(x) - x) < \varepsilon$. It follows that $p(R(x) - x) \leq \theta(x)\, p(s(x) - x) + \Sigma_K \theta_K(x)\, p(r_K(x) - x) < \varepsilon$. Q.E.D.

COROLLARY 1. - Let Y be a closed, metrizable subset of an lctvs X. Let $C \subset Y$ be a finite union of closed, convex sets C_i, $C = \bigcup_{i=1}^m C_i$. Then there exists a compact, finite dimensional (i.e., its range lies in a finite dimensional subspace of X) map $R : C \to C$ such that $R(x) \in C_i$ if $x \in C_i$.

PROOF. - For each $J \subset \{1, 2, \ldots, m\}$ such that C_J is nonempty, select $x_J \in C_J$. For $1 \leq j \leq m$, let $D_j = \mathrm{co}\{x_J | j \in J\}$, where $\mathrm{co}(S)$ denotes the convex hull of a set S. Since D_j is the continuous image of the compact set of ordered N-tuples $\{(\lambda_L) | L \subset \{1, 2, \ldots, m\}, C_L \neq \emptyset, j \in L, \sum_L \lambda_L = 1\}$ (where N denotes the number of subsets $L \subset \{1, 2, \ldots, m\}$ such that $j \in L$ and C_L is nonempty) under the map $(\lambda_L) \to \sum_L \lambda_L x_L$, D_J is compact. It is clear that $D_j \subset C_j$ and that $\bigcap_{j \in J} D_j$ is nonempty iff $\bigcap_{j \in J} C_j$ is nonempty for all $J \subset \{1, 2, \ldots, m\}$. It follows by Theorem 1 that there exists a retraction $R : C \to D = \bigcup_{j=1}^m D_j$ and $R(x) \in C_j$ if $x \in C_j$. R is obviously compact and finite dimensional. Q.E.D.

COROLLARY 2. - Let Y be a closed, completely metrizable subset of an lctvs X. Let $A_n = \bigcup_{i=1}^m C_{i,n}$ be a union of m bounded, closed, convex subsets of Y, m independent of n. Assume that $C_{i,n} \supset C_{i,n+1}$ for $1 \leq i \leq m$, $1 \leq n \leq \infty$ and suppose that $\gamma(A_n) \to 0$, where γ denotes the measure of noncompactness with respect to a complete metric d for Y. Let $A_\infty = \bigcap_{n \geq 1} A_n$. Then there exists $n_0 \geq 1$ such that for $n \geq n_0$ there exists a retraction $R_n : A_n \to A_\infty$ such that $R_n(x) \in C_{i,n}$ if $x \in C_{i,n}$. If p is any continuous seminorm and $\varepsilon > 0$, there exists an integer n_1 such that for $n \geq n_1$ the R_n can be chosen so that $p(R_n(x) - x) < \varepsilon$.

PROOF. - If we write $C_{i,\infty} = \bigcap_{n \geq 1} C_{i,n}$, it is easy to see that $A_\infty = \bigcup_{i=1}^m C_{i,\infty}$. Also, if for $J \subset \{1, 2, \ldots, m\}$ we put $C_{J,\infty} = \bigcap_{j \in J} C_{j,\infty}$ and $C_{J,n} = \bigcap_{j \in J} C_{j,n}$, $C_{J,\infty} = \bigcap_{n \geq 1} C_{J,n}$. In order to prove the first part of the Corollary, it suffices then $\mathrm{cl}(U_K \cap C_j$ is empty. But $U_K \subset V_K$, so $\mathrm{cl}(U_K) \cap C_j \subset \mathrm{cl}(V_K) \cap C_j$, which is empty.

This completes the inductive step. After m steps we obtain the desired covering. Q.E.D.

We can now prove our main theorem.

THEOREM 1. - Let Y be a closed, metrizable subset of an lctvs X. Let $C \subset Y$ be a finite union of m closed, convex sets C_i, $C = \bigcup_{i=1}^{m} C_i$ and let $D = \bigcup_{i=1}^{m} D_i$ be a finite union of m closed, convex sets D_i such that $D_i \subset C_i$. For every subset $J \subset \{1, 2, \ldots, m\}$ assume that $D_J \equiv \bigcap_{j \in J} D_j$ is empty if and only if $C_J \equiv \bigcap_{j \in J} C_j$ is empty. Then there exists a retraction R of C onto D such that $R(x) \in C_i \cap D$ if $x \in C_i$, $1 \leq i \leq m$. (In particular, D is a deformation retract of C by the deformation retraction $H(x, t) = (1-t)R(x) + tx$, $0 \leq t \leq 1$). Furthermore, let p be a continuous seminorm on X and ε a positive number. Then there exist open neighborhoods $\tilde{O}_K$ in Y of D_K for $K \subset \{1, 2, \ldots, m\}$ ($\tilde{O}_K$ is empty if D_K is empty) such that if $C_K \subset \tilde{O}_K$ for all K, R can be choosen so that $p(Rx - x) < \varepsilon$ for $x \in C$.

PROOF. - For each nonempty D_K, let $r_K : Y \to D_K$ be a retraction of Y onto D_K. Such retractions exist since D_K is an AR [19]. Let $\tilde{O}_K = \{x \in Y \mid p(r_K(x) - x) < \varepsilon\}$ if D_K is nonempty and $\tilde{O}_K$ be empty otherwise. If $C_K \subset \tilde{O}_K$ for all K, let $O_K = \tilde{O}_K \cap C$. Otherwise, let O_K be any open neighborhood of C_K in C such that O_K is empty if C_K is empty. By Lemma 2 there exists an open covering $\{U_K\}$ of C such that: (1) $U_K \subset O_K$; (2). If $|L| \geq |K|$ but $L \not\supset K$, $U_L \cap U_K$ is empty; (3). If $j \notin K$, $\mathrm{cl}(U_L) \cap C_j$ is empty. By Property 3 of the covering it is clear that for all $L \subset \{1, 2, \ldots, m\}$ there exists an open neighborhood W_L of $\mathrm{cl}(U_L)$ such that $W_L \cap C_j$ is empty for $j \notin L$. By Lemma 1 there exists an open neighborhood V of D in C and a retraction $s : V \to D$ such that $s : V \cap C_i \to D \cap C_i$. Let $W = \{x \in V \mid a)\ p(s(x) - x) < \varepsilon$ and b). If $x \in \mathrm{cl}(U_L)$ for any $L \subset \{1, 2, \ldots, m\}$, then $s(x) \in W_L\}$. Since $s(x) = x$ for $x \in D$, it is clear thet $D \subset W$; and it is not hard to see that W is open. Thus W is an open neighborhood of D. Let W_1 be a closed neighborhood of D such that $W_1 \subset W$ and denote by W_1' the complement of W_1 in C. Consider the open covering of C by $\{W, U_K \cap W_1'\}$ and let $\{\theta, \theta_K\}$ denote a partition of unity subordinate to this covering, i.e., $\mathrm{supp}(\theta) \subset W$, $\mathrm{supp}(\theta_K) \subset U_K \cap W_1'$ and $\theta(x) + \sum_K \theta_K(x) = 1$ for $x \in C$. We define $R(x) = \theta(x)s(x) + \sum_K \theta_K(x) r_K(x)$; the summation, of course, is only over K such that D_K is nonempty and $\theta(x)s(x)$ is defined to be 0 for $x \notin V$.

We have to show that R satisfies the claims of the theorem. First, let us prove that if $x \in C_j$, $R(x) \in C_j$. If $j \notin L \subset \{1, 2, \ldots, m\}$, we know that $U_L \cap C_j$ is empty. Thus $\theta_L(x) = 0$ unless $j \in L$, and for such L, $r_L(x) \in D_L \subset C_j$. If $x \in V$, so that $s(x)$ is defined, $s(x) \in C_j$. Thus $R(x) = \theta(x)s(x) + \sum_K \theta_K(s) s_K(x)$ is a convex combination of points in C_j and hence lies in C_j.

LEMMA 2. - Let Y be a closed, metrizable subset of an lctvs X. Let $C \subset Y$ be a finite union of closed, convex sets C_i, $C = U_{i=1}^{m} C_i$. For each subset $J \subset \{1, 2, \ldots, m\}$, let O_J denote an open neighborhood of $C_J = \bigcap_{j \in J} C_j$ if C_J is nonempty and let O_J denote the empty set if C_J is empty. Then there exists an open covering $\{U_J\}$ of C (indexed by subsets $J \subset \{1, 2, \ldots, m\}$) which satisfies the following properties:

(1) $U_J \subset O_J$.

(2) If $|K| \geq |J|$ but $K \not\supset J$, $U_K \cap U_J$ is empty.

(3) If $j \notin K$, $\mathrm{cl}(U_K) \cap C_j$ is empty.

PROOF. - We shall construct $\{U_K\}$ inductively, starting with $|K| = m$, then considering $|K| = m-1$, and so on. For $|K| = m$, so that $K = \{1, 2, \ldots, m\}$, we define $U_K = O_K$. Notice that $U_{|K| \geq m} U_k \supset U_{|K| \geq m} C_K$ and that $\{U_K \,|\, |K| \geq m\}$ trivially satisfies conditions (1)-(3).

Assume that we have constructed open sets U_L in C for $|L| \geq r$, $m \geq r > 1$, such that $U_{|L| \geq r} U_L \supset U_{|L| \geq r} C_L$ and $\{U_L \,|\, |L| \geq r\}$ satisfies conditions (1)-(3) above. Select $K \subset \{1, 2, \ldots, m\}$ with $|K| = r - 1$. We wish to define U_K. Let $A_K = C_K \sim U_{|L| \geq r} U_L$ and notice that for $j \notin K$, A_K and C_j are disjoint closed sets. Thus there exists an open neighborhood V_K of A_K such that $V_K \subset O_K$ and $\mathrm{cl}(V_K) \cap C_j$ is empty for $j \notin K$. Consider all $L \subset \{1, 2, \ldots, m\}$ such that $|L| > |K| = r - 1$ but $L \not\supset K$, and for each such L, select $j \in K$ such that $j \notin L$. If (L, j) is such a pair, we know by the inductive hypothesis that $\mathrm{cl}(U_L) \cap C_j$ is empty, so there exists an open neighborhood $W_{(L,j)}$ of C_j such that $\mathrm{cl}(U_L) \cap W_{L,j}$ is empty. Since $C_j \supset C_K \supset A_K$, $W_{(L,j)}$ is an open neighborhood of A_K. Let us set $W_K = \bigcap_{(L,j)} W_{(L,j)}$, where the intersection is taken over all pairs (L, j) such that $|L| > |K|$, $L \not\supset K$, and $j \in K$ but $j \notin L$. Finally, we note that for K, $K' \subset \{1, 2, \ldots, m\}$, $|K| = |K'| = r - 1$, $K = K'$, $A_K \cap A_{K'}$ is empty, since $A_K \cap A_{K'} \subset C_{K \cup K'}$, and $|K \cup K'| \geq r$. Thus it is not hard to see that we can find open neighborhood Z_K of A_K for $|K| = r - 1$ such that for any two unequal subsets K and K' with $|K| = |K'| = r - 1$, $Z_K \cap Z_{K'}$ is empty. We define $U_K = V_K \cap W_K \cap Z_K$, and we have to show that $\{U_K \,|\, |K| \geq r - 1\}$ satisfies the inductive hypotheses.

Since U_K, $|K| = r - 1$, is an open neighborhood of A_K, we clearly have $U_{|K| \geq r-1} C_K \subset (U_{|K| \geq r} U_K) \cup (U_{|K| = r-1} A_K) \subset (U_{|K| \geq r-1} U_K)$. We selected $V_K \subset O_K$, so the first condition on the covering is satisfied. To check the second condition, we must show that if $L \subset \{1, 2, \ldots, m\}$, $|L| \geq |K| = r - 1$, and $L \not\supset K$, then $U_L \cap U_K$ is empty. If $|L| > |K|$, select $j \in K$, such that $j \notin L$. By construction we have $U_L \cap U_K \subset U_L \cap W_{(L,j)}$, which is empty. If $|L| = |K|$, we have $U_L \cap U_K \subset Z_L \cap Z_K$, which is empty. Finally, to check the third condition, we have to show that if $K \subset \{1, 2, \ldots, m\}$, $|K| = r - 1$ and $j \notin K$, of closed, convex sets contained in a metrizable subset of an lctvs, so that $C_K \cap D$ is an ANR (see Section C). Restricting our attention to C_K, $|K| = m$, we thus see that there exists a closed neighborhood U_K in C_K of $C_K \cap D$ and a retraction $s_K : U_K \to C_K \cap D$. Let us write $s_m = i \cup s_K$ and $Z_m = C \cup U_K$, $|K| = m$. It is clear that s_m is a retraction of Z_m onto D and that $s_m(x) \in C_j$ if $x \in C_j \cap Z_m$.

Now let us proceed inductively to find s and V. Suppose we have found closed neighborhoods U_L in C_L of $C_L \cap D$ for all L such that $|L| = p$, $m \geq p > 1$, and suppose we also have a retraction $s_p : Z_p = D \cup \cup_{|L|=p} U_L \to D$ such that $s_p : Z_p \cap C_j \to C_j$ for $1 \leq j \leq m$. Consider C_K, $|K| = p-1$, and notice that by our assumptions $s_p : Z_p \cap C_K \to D \cap C_K$. Since $Z_p \cap C_K$ is a closed subset of the metric space C_K and since $D \cap C_K$, a finite union of closed, convex, metrizable sets, is an ANR, there exists an extension of the map $s_p : Z_p \cap C_K \to D \cap C_K$ to an open neighborhood U'_K of $Z_p \cap C_K$ in C_K. Let us call this extended map s_K, so $s_K : U'_K \to D \cap C_K$ is a retraction.

It is necessary to consider a smaller neighborhood than U'_K. Notice that $U_{|L|=p} U_L$ is a neighborhood in $U_{|L|=p} C_L$ of $D \cap (U_{|L|=p} C_L)$. Since $U_{|L|=p} C_L$ is closed in C, $C_K \sim U_{|L|=p} C_L$ is open in C_K. Thus it is not difficult to see that $W_K \equiv (C_K \sim U_{|L|=p} C_L) \cup (U_{|L|=p} (U_L \cap C_K))$ is a neighborhood in C_K of $C_K \cap D$. This implies that $U'_K \cap W_K$ is a neighborhood in C_K of $C_K \cap D$. We take U_K to be a closed neighborhood of $C_K \cap D$ in C_K such that $U_K \subset U'_K \cap W_K$, and we define $Z_{p-1} = D \cup U_{|K|=p-1} U_K$ and $s_{p-1} = s_p \cup (U_{|K|=p-1} (s_K | U_K))$.

Our first claim is that this definition actually gives a well-defined retraction of Z_{p-1} onto D. To see this it suffices to show that if $|K| = |K'| = p-1$, $K \neq K'$, and $x \in U_K \cap U_{K'}$, then $s_K(x) = s_{K'}(x)$. But $U_K \cap U_{K'} \subset C_{K \cup K'}$ and $|K \cup K'| \geq p$, so it follows, since $U_K \subset W_K$, that $x \in U_J$ for some J with $|J| = p$. Since all the s_K reduce to s_p on U_J, $|J| = p$, $s_K(x) = s_p(x) = s_{K'}(x)$.

Next we have to show that if $x \in Z_{p-1} \cap C_j$, $s_{p-1}(x) \in C_j$, $1 \leq j \leq m$. This condition is certainly satisfied if $x \in D$, so suppose that $x \in U_K \cap C_j$, $|K| = p-1$. If it is also true that $x \in U_L$ for some L such that $|L| = p$, then by our construction $s_{p-1}(x) = s_p(x)$ and by inductive hypothesis $s_p(x) \in C_j$. If $x \in U_K \cap C_j$ but $x \notin U_{|L|=p} U_L$ we know by the construction of U_K that $x \in C_K \sim U_{|L|=p} C_L$. But we also know that $x \in U_K \cap C_j \subset C_{K \cup \{j\}}$, so we must have that $j \in K$ – otherwise $|K \cup \{j\}| \geq p$. It follows that $C_K \subset C_j$ in this case, and since wo constructructed s_{p-1} so that $s_{p-1} : U_K \to C_K \cap D \subset C_j$, $s_{p-1}(x) \in C_j$.

This completes the inductive step. After carrying through the above construction m times, we obtain $s_m = s$ and Z_m, which contains an open neighborhood V of D. Q.E.D.

Our next lemma is somewhat artificial, but it will prove crucial in establishing our theorems.

number lim sup $(\gamma(A^n))^{1/n}$ is studied in [29] and in the author's dissertation. (University of Chicago, 1969) where it is proved that lim sup $(\gamma(A^n))^{1/n} =$ sup $\{|\lambda| : \lambda \in \text{ess}(A)\}$. Ess (A) denotes the essential spectrum of A as defined by BROWDER [7].

B. - Geometrical properties of finite unions of convex sets.

In this section we shall establish theorems concerning the geometric properties of finite unions of closed, convex sets in a locally convex topological vector space. These results will prove crucial for the theory of the fixed point index for k-set-contractions.

Let us begin by recalling some basic facts. We shall say that a topological space X is an absolute neighborhood retract (abbreviated ANR) if given any metric space M, a closed subspace $A \subset M$, and a continuous map $f : A \to X$, there exists an open neighborhood U of A and a continuous map $F : U \to X$ such that $F(a) = f(a)$ for $a \in A$; X is called an absolute retract (AR) if F as above can be defined on all of M. A theorem of DUGUNDJI [15] asserts that any convex subset of a locally convex topological space is an AR. A classical result says that if X_1 and X_2 are subspaces of a topological space Y and X_1, X_2, and $X_1 \cap X_2$ are ANR's, then $X_1 \cup X_2$ is an ANR. This implies immediately that if C_i, $1 \leq i \leq m$, are convex subsets of some lctvs X, then $C = U_{i=1}^{m} C_i$ is an ANR. It follows that if we assume C_i, $1 \leq i \leq m$, is a closed. convex subset of a metrizable subset Y of a lctvs X, then there exists an open neighborhood U in Y of $C = U_{i=1}^{n} C_i$ and a retraction $R : U \to C$, simply because the identity map $i : C \to C$ can be extended as a map from an open neigborhood of C to C in this case. It is this fact which we shall use repeatedly.

With the aid of these results we can establish an essential lemma.

LEMMA 1. - Let Y be a closed, metrizable subset of an lctvs X. Let $C \subset Y$ be a finite union of m closed, convex sets C_i, $C = U_{i=1}^{m} C_i$, and let $D \subset C$ be a finite union of closed, convex sets $D_i \subset C_i$, $D = U_{i=1}^{m} D_i$. Then there exists in C an open neighborhood V of D and a retraction $s : V \to D$ such that $s(x) \in C_i$ if $x \in C_i$.

PROOF. - First, let us establish some notation. We shall denote by J, K, L, M subsets of $\{1, 2, \ldots, m\}$ and by $|J|$ the number of elements of J. We shall write $C_J = \cap_{j \in J} C_j$ and $D_J = \cap_{j \in J} D_j$.

We begin with our map s defined and equal to the identity i on D, and we want to extend s. First, let us extend s to a closed neighborhood U_K in C_K of $C_K \cap D$, $|K| = m$. (We allow the possibility that C_K is empty, in which case U_K is taken to be empty). Notice that $C_K \cap D$ is a finite union

The apparatus of measure of noncompactness and k-set-contractions can be used to establish fixed point theorems. The proof of the following proposition is due to DARBO, but DARBO stated the proposition in less generality.

PROPOSITION 10. - Let C be a closed, bounded, convex set in a BANACH space X. Let $f: C \to C$ be a continuous map. Let $C_1 = \overline{\text{co}}\, f(C)$ and $C_n = \overline{\text{co}}\, f(C_{n-1})$ for $n > 1$. Assume that $\gamma(C_n) \to 0$. Then f has a fixed point.

PROOF. - It is clear that C_n is closed, bounded, convex, and nonempty and $C_n \supset C_{n+1}$ for $n \geq 1$. By Proposition 2 $C_\infty = \bigcap_{n\geq 1} C_n$ is nonempty and compact, and C_∞ is certainly convex. By our construction $f: C_n \to C_{n+1}$ so that $f: C_\infty \to C_\infty$. It follows by Schauder's fixed point theorem that f has a fixed point. Q.E.D.

COROLLARY 2. - (DARBO, [12]). Let C be a closed, bounded, convex set and $f: C \to C$ a k-set-contraction, $k < 1$. Then f has a fixed point.

PROOF. - It suffices to show $\gamma(C_n) \to 0$. But $\gamma(C_1) = \gamma(\overline{\text{co}}\, f(C)) = \gamma(f(C)) \leq$ $\leq k\gamma(C)$, and generally $\gamma(C_n) = \gamma(\overline{\text{co}}\, f(C_{n-1})) = \gamma(f(C_{n-1})) \leq k\gamma(C_{n-1})$. This implies that $\gamma(C_n) \leq k^n \gamma(C)$. Q.E.D.

COROLLARY 3. - Let A be a bounded linear operator in a BANACH space X and assume $\limsup \gamma(A^n)^{1/n} = k < 1$. Let C be a closed, bounded, convex set in X and $B: C \to X$ a compact (not necessarily linear) map. Assume that if $f(x) = A(x) + B(x)$, $f: C \mapsto C$. Then f has a fixed point.

PROOF. - In the notation of Proposition 10 it suffices to show $\gamma(C_n) \to 0$. To see this first notice that for $S \subset C$, $\gamma(f(\overline{\text{co}}\, S)) = \gamma(f(S))$. We have $\gamma(f(\overline{\text{co}}\, S)) \geq \gamma(f(S))$, since $\overline{\text{co}}\, S \supset S$. On the other hand $f(\overline{\text{co}}\, S) \subset A(\overline{\text{co}}\, S) +$ $+ B(\overline{\text{co}}\, S)$, so by Proposition 3, $\gamma(f(\overline{\text{co}}\, S)) \leq \gamma(A(\overline{\text{co}}\, S)) + \gamma(B(\overline{\text{co}}\, S)) =$ since B is compact) $\gamma(A(\overline{\text{co}}\, S))$. Since A is linear it follows that $A(\overline{\text{co}}\, S) = \overline{\text{co}}\, A(S)$, so $\gamma(A(\overline{\text{co}}\, S)) \leq \gamma(\overline{\text{co}}\, A(S)) = \gamma(A(S))$. However, $A(x) = f(x) - B(x)$ so $A(S) \subset$ $\subset f(S) - B(S) = \{f(s) - B(s) \mid s \in S\}$. This implies that $\gamma(A(S)) \leq \gamma f(S)) +$ $+ \gamma(-B(S)) = \gamma(f(S))$, and consequently $\gamma(f(\overline{\text{co}}\, S)) \leq \gamma(f(S))$.

Notice that $f^j = A^j + B_j$, where $B_j: C \to X$ is a compact map and A^j is, of course, a bounded linear map. Thus the above reasoning implies that $\gamma(f^j(\overline{\text{co}}\, S)) = \gamma(f^j(S))$ for $S \subset C$. Applying this relation repeatedly we see that $\gamma(C_n) = \gamma(f(C_{n-1})) = \gamma(f(\overline{\text{co}}\, f(C_{n-1}))) = \gamma(f^2(C_{n-2})) = \ldots = \gamma(f^n(C))$. However, $f^n =$ $= A^n + B_n$, where B_n is a compact, so $\gamma(f^n(C)) \leq \gamma(A^n(C)) + \gamma(B_n(C)) = \gamma(A^n(C))$. Since $\limsup (\gamma(A^n))^{1/n} = k < 1$, for $n \geq n_0$, $\gamma(A^n(C)) \leq k^n \gamma(C) \to 0$. Q.E.D.

Corollary 3 was proved by NASHED and WONG [27] under the stronger assumptions that 1) $Ax + By \in C$ for all $x, y \in C$ and 2) $\lim_{n\to\infty} \|A^n\|^{1/n} < 1$. The

If $f, g \in C(A, X_2)$, $d(f, g)$ is, by definition, $\sup_{x \in A} d_2(f(x), g(x))$. Assume that for $y \in A$, the map $y \to V(\cdot, y)$ form A to $C(A, X_2)$ is compact on bounded sets.

PROPOSITION 8. - Let $V: X_1 \times X_1 \to X_2$ be as above and let $f(x) = V(x, x)$. Then f is a k-set-contraction.

PROOF. - Let A be a bounded subset of X_1. Since the map $y \to V(\cdot, y)$ is compact from A to $C(A, X_2)$, we can select $y_1, \ldots, y_n \in A$, such that for $y \in A$, there exists i such that $\sup_{x \in A} d_2(V(x, y), V(x, y_i)) < \varepsilon$. (This is just the translation of the fact that $\{V(\cdot, y)\} \subset C(A, X_2)$ is totally bounded in $C(A, X_2)$). If we write $V(\cdot, y) = V_y$, this means that $f(A) \subset V(A \times A) \subset \bigcup_{i=1}^{n} N_\varepsilon(V_{y_i}(A))$. It follows that $\gamma_2(f(A)) \leq \max_{1 \leq i \leq n} \{\gamma_2(V_{y_i}(A)) + 2\varepsilon\}$. But since V_{y_i} is a k-contraction, $\gamma_2(V_{y_i}(A)) \leq k\gamma_1(A)$, so $\gamma_2(f(A)) \leq k\gamma_1(A) + 2\varepsilon$, whence (since ε can be taken as small as desired) f is a k-set-contraction. Q.E.D.

Mappings like V are considered in [10], and we also encounter similar maps in applications.

Our next proposition indicates another way in which k-set-contractions may arise.

PROPOSITION 9. - Let (X, d) be a metric space and $\{0_i \mid 1 \leq i \leq n\}$ be a finite open covering. Let Y be a BANACH space and $f_i : 0_i \to Y$ a k-set-contraction, $1 \leq i \leq n$. Suppose that $\{\lambda_i\}$ is a partition of unit subordinate to the open covering $\{0_i\}$. Then the map $g(x) = \sum_{i=1}^{n} \lambda_i(x) f_i(x)$ is a k-set-contraction.

PROOF. - Let A be a bounded subset of X. Then since $\sum_{i=1}^{n} \lambda_i(x) = 1$ for every $x \in A$, it is clear that $g(A) \subset \mathrm{co}\,[\bigcup_{i=1}^{n} f_i(A \cap 0_i)]$. It follows that $\gamma(g(A)) \leq$ $\leq \gamma(\mathrm{co}\,(\bigcup_{i=1}^{n} f_i(A \cap 0_i))) = \gamma(\bigcup_{i=1}^{n} f_i(A \cap 0_i)) = \max_{1 \leq i \leq n} \gamma(f_i(A \cap 0_i)) \leq k\gamma(A)$. Q.E.D.

COROLLARY 1. - Let $B = \{x : \|x\| \leq 1\}$ in a BANACH space X and let $R: X \to B$ be the radial projection, i.e. $R(x) = \frac{x}{\|x\|}$ for $\|x\| \geq 1$ and $R(x) = x$ for $\|x\| \leq 1$. Then R is a 1-set-contraction.

PROOF. - Let $f_1(x) = x$, $f_2(x) = 0$, $\lambda_1(x) = \frac{1}{\|x\|}$ for $\|x\| \geq 1$ and $\lambda_1(x) = 1$ for $\|x\| \leq 1$, and $\lambda_2(x) = 1 - \lambda_1(x)$. Then $R(x) = \lambda_1(x) f_1(x) + \lambda_2(x) f_2(x)$, so Proposition 9 implies the result. Q.E.D.

Corollary 1 is interesting, for as de FIGUREIDO and KARLOVITZ have shown [16], if dimension $(X) \geq 3$, R is a 1-contraction if and only if X is a HILBERT space.

b) Let A be a bounded set in X. Then $(f+g)(A) \subset f(A) + g(A)$, so by Proposition 3, $\gamma((f+g)(A)) \leq \gamma(f(A) + g(A)) \leq \gamma(f(A)) + \gamma(g(A)) \leq k_1\gamma(A) + k_2\gamma(A) = (k_1 + k_2)\gamma(A)$, so $f+g$ is a (k_1+k_2)-set-contraction. Q.E.D.

Notice that in the above we used γ to refer to the measure of noncompactness in X and in Y, even though they are different. We shall occasionally do this if it seems no confusion will result.

Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f: X_1 \to X_2$ a continuous map. We shall say that f is compact on bounded sets or, occasionally, simply compact, if given a bounded set $A \subset X_1$, $f(A)$ has compact closure. We shall say that f is a k-contraction if $d_2(f(x), f(y)) \leq kd_1(x, y)$ for every $x, y \in X_1$. We emphasize that f being a k-contraction is much less general than its being a k-set-contraction.

PROPOSITION 7. (DARBO). - Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f: X_1 \to X_2$ a continuous map. a) If f is a k-contraction, then f is a k-set-contraction. b) If f is compact on bounded sets, then f is a 0-set-contraction. Conversely, if X_2 is complete and f is a 0-set-contraction, then f is compact on bounded sets.

PROOF. - a) Let A be a bounded set in X_1 and suppose $\gamma_1(A) = d$. Then given $\varepsilon > 0$, we can write $A = \bigcup_{j=1}^{m} S_j$, $\operatorname{diam}(S_j) \leq d + \varepsilon$. Thus $f(A) = \bigcup_{j=1}^{m} f(S_j)$, and since f is a k-contraction, $\operatorname{diam}(f(S_j)) \leq k(d+\varepsilon)$. Since ε is arbitrary, $\gamma_2(f(A)) \leq kd$ and f is a k-set-contraction.

b) Let A be a bounded set in X_1. Since we are assuming f is compact on bounded sets, $clf(A)$ is compact and hence totally bounded. Thus $\gamma_2(clf(A)) = \gamma_2(f(A)) = 0$, so f is a 0-set-contraction.

Conversely, if X_2 is complete and f is a 0-set-contraction, then for any bounded A, $\gamma_2(clf(A)) = \gamma_2(f(A)) = 0$. This means $clf(A)$ is totally bounded, and since X_2 is complete, $clf(A)$ is compact, i.e., f is compact on bounded sets. Q.E.D.

Using Proposition 6 and 7 we can construct examples of k-set-contractions. For instance, let G be a subset of a BANACH space X, $U: G \to X$ a k-contraction and $C: G \to X$ compact on bounded sets. Then $U + C$ is a k-set-contraction.

In our next proposition we want to generalize the above example. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $V: X_1 \times X_1 \to X_2$ be a continuous map. Assume that for $y \in X_1$, $V(\cdot, y): X_1 \to X_2$ is a k-contraction, k independent of y. Let A be any bounded subset of X_1 and denote by $C(A, X_2)$ the metric space of bounded, continuous functions from A to X_2.

is bounded and $\gamma_2(f(A)) \leq k\gamma_1(A)$. Of course γ_i denotes the measure of noncompactness in X_i, $i = 1, 2$. If f is a k-set-contraction, we define $\gamma(f) = \inf\{k \geq 0 \mid f$ is a k-set-contraction$\}$. Given $f : X_1 \to X_2$, we shall say that f is a local strict-set-contraction if for all $x \in X_1$ there exists a neigborhood N_x of x such that $f|N_x$ is a k_x-set-contraction, $k_x < 1$.

We shall also need a slight generalization of k-set-contractions, $k < 1$, essentially due to B. N. SADOVSKII [36] (SADOVSKII actually used a different measure of noncompactness). Given a continuous map $f : X_1 \to X_2$ we say that f is a condensing map if for every bounded set A in X_1 such that $\gamma_1(A) \neq 0$, $\gamma_2(f(A)) < \gamma_1(A)$. We say that f is a local condensing map if every $x \in X_1$, has a neighborhood N_x such that $f|N_x$ is a condensing map.

Of course every k-set-contraction with $k < 1$ is a condensing map, but the converse is not true. To see this, let $\rho : [0, 1] \to \mathbb{R}$ be a strictly decreasing, nonnegative function such that $\rho(0) = 1$. Let B denote the unit ball about 0 in an infinite dimensional BANACH space X and consider the map $f : B \to B$ given by $f(x) = \rho(\|x\|)x$. We claim that f is a condensing map, but that f is not a k-set-contraction for any $k < 1$. To see that f is not a k-set-contraction for $k < 1$, consider $f(V_r(0)$ for $0 < r \leq 1$, where $V_r(0)$ is the closed ball of radius r. It is easy to see that $f(V_r(0)) \supset V_{\rho(r)r}(0)$. By Proposition 5, $\gamma(V_{\rho(r)r}(0)) = 2\rho(r)r$ and since $\gamma(V_r(0)) = 2r$, it follows that at best f is a $\rho(r)$-set-contraction. Since $\rho(r) \to 1$ as $r \to 0$, f cannot be a k-set-contraction for any $k < 1$. On the other hand, f is a 1-set-contraction, for if A is any subset of B, $f(A) \subset \overline{\mathrm{co}}\,\{A \cup \{0\}\}$ and thus $\gamma f(A) \leq \gamma(A \cup \{0\}) = \gamma(A)$. However, we can say more than this. Suppose $A \subset B$ and $\gamma(A) = d > 0$. Select $r < d/2$, define $A_1 = A \cap V_r(0)$, $A_2 = A \cap V_r'(0)$ ($V_r'(0) =$ complement of $V_r(0)$), and consider $f(A) = f(A_1) \cup f(A_2)$. Since f is a 1-set-contraction, $\gamma f(A_1)) \leq 2r < d = \gamma(A)$. Since ρ is strictly decreasing and $\|x\| \geq r$ for $x \in A_2$, $f(A_2) \subset \{sa \mid 0 \leq s \leq \rho(r), a \in A_2\} \subset \overline{\mathrm{co}}\,\{\rho(r)A \cup \{0\}\}$ and $\gamma(f(A_2)) \leq \rho(r)\gamma(A) < \gamma(A)$. It follows that $\gamma(f(A)) = \max\{\gamma(f(A_1)), \gamma(f(A_2))\} < \gamma(A)$, so that f is a condensing map. The same proof also shows that f is not a local strict-set-contraction.

We now wish to state the elementary properties of k-set-contractions and give some examples which indicate their usefulness.

PROPOSITION 6. (DARBO) a). - Let (X_i, d_i), $i = 1, 2, 3$ be metric spaces. Assume that $f : X_1 \to X_2$ is a k_1-set-contraction and $g : X_2 \to X_3$ is a k_2-set-contraction. Then gf is a k_1k_2-set-contraction. b) Let (X, d) be a metric space and Y a BANACH space. Assume that $f : X \to Y$ is a k_1 set-contraction and $g : X \to Y$ is a k_2-set-contraction. Then $f + g$ is a $(k_1 + k_2)$-contraction.

PROOF. - a) Let A be a bounded set in X_1. Then $f(A)$ is a bounded set in X_2 and $\gamma_2(f(A)) \leq k_1\gamma_1(A)$. Since g is a k_2-set-contraction, $\gamma_3(g(f(A))) \leq k_2\gamma_2(f(A)) \leq k_1k_2\gamma_1(A)$, i.e., gf is a k_1k_2-set-contoaction.

PROPOSITION 3. (DARBO). - Let X be a BANACH space and suppose A and B are bounded subsets of X. If we denote $\{a+b \mid a \in A, b \in B\}$ by $A+B$, then $\gamma(A+B) \leq \gamma(A)+\gamma(B)$.

PROPOSITION 4. (DARBO). - Let X be a BANACH space and A a bounded subset of X. If we denote the convex closure of A by $\operatorname{cocl}(A)$ (we may also write $\overline{\operatorname{co}}(A)$ for the convex closure of A), $\gamma(\operatorname{cocl}(A))=\gamma(A)$.

In general, given a subset A of a BANACH space X, there is no easy algorithm to determine $\gamma(A)$. However, if A is a ball in an infinite dimensional BANACH space, we can describe $\gamma(A)$.

M. FURI and A. VIGNOLI have also obtained this result [40], but we believe it was first proved in our dissertation [31].

PROPOSITION 5. - Let X be an infinite dimensional BANACH space, let $B=\{x \mid \|x\| \leq 1\}$ and $S=\{x \mid \|x\|=1\}$. Then $\gamma(B)=\gamma(S)=2$.

PROOF. - Since $B=\operatorname{cocl}(S)$, Proposition 4 implies that $\gamma(B)=\gamma(S)$, and since $\operatorname{diam}(S)=2$, it is certainly true that $\gamma(S) \leq 2$. To see that $\gamma(S)=2$, we proceed by contradiction. If $\gamma(S)<2$, we can write $S=\bigcup_{j=1}^{n} T_j$, where $\operatorname{diam}(T_j)<2$, and by taking $\operatorname{cl}(T_j)$ instead of T_j if necessary, we can assume T_j is closed. Let F be an n dimensional subspace of X and consider $S \cap F=\bigcup_{j=1}^{n}(T_j \cap F)$. By the LUSTERNIK-SCHNIRELMAN-BORSUK theorem [17, p. 50], if the unit sphere (with respect to any norm) in an n-dimensional vector space V is covered by n closed sets, then at least one of the sets contains a pair of antipodal points, i.e., points x and $-x$ for some x on the unit sphere. In our case this means some $T_j \cap F$ contains a pair of antipodal points, so for this $T_j \cap F$, $2 \leq \operatorname{diam}(T_j \cap F) \leq \operatorname{diam}(T_j)$, a contradiction.
Q.E.D.

We should remark that the LUSTERNIK-SCHNIRELMAN-BORSUK theorem is proved in [17] for a particular norm on R^n, but since all norms on R^n are equivalent it is easy to see that the unit spheres with respect to different norms are homeomorphic with a homeomorphism which takes antipodal points to antipodal points. This shows the theorem true for any norm on R^n. For an arbitrary n dimensional vector space V, we just select an isomorphism $T: V \rightarrow R^n$ and take a norm on R^n such that the isomorphism is an isometry. The theorem for V follows.

Closely associated with the notion of measure of noncompactness is the concept of «k-set-contractions» (also due to KURATOWSKI, [20]). Let (X_1, d_1) and (X_2, d_2) be metric spaces and suppose $f: X_1 \rightarrow X_2$ is a continuous map. We say that f is a k-set-contraction if given any bounded set A in X_1, $f(A)$

Assume that $\gamma(A_n)$ converges to 0. Then if we write $A_\infty = \bigcap_{n\geq 1} A_n$, A_∞ is a nonempty compact set and A_n approaches A_∞ in the HAUSDORFF metric.

PROOF. - Since $\gamma(A_\infty) \leq \gamma(A_n)$ for all n, it is clear that $\gamma(A_\infty) = 0$, and since A_∞ is an intersection of closed sets it is closed. It follows that A_∞ is compact. We have to show A_∞ is nonempty.

Let us write $\gamma(A_n) = d_n$ and select a sequence of positive numbers ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$. By assumption we can write $A_n = \bigcup_{j=1}^{k(n)} S_{j,n}$, where diam $(S_{j,n}) \leq d_n + \varepsilon_n$, $1 \leq j \leq k(n)$. We know that for some j, $1 \leq j \leq k(1)$, $S_{j,1} \cap A_n$ is nonempty for all n. Otherwise, we would have $A_n = A_1 \cap A_n = (\bigcup_{j=1}^{k(1)} S_{j,1}) \cap A_n$ empty for n large enough, a contradiction. Let us denote by T_1 the $S_{j,1}$ such that $S_{j,1} \cap A_n \neq \emptyset$ for all n.

We now proceed by induction. Assume we have found $T_1, T_2, \ldots, T_m$ such that $T_i = S_{j,i}$ for some j, $1 \leq j \leq k(i)$, and such that $(\bigcap_{i=1}^{m} T_i) \cap A_n$ is is nonempty for all n. Let us write $T = \bigcap_{i=1}^{m} T_i$. We want to find $T_{m+1} = S_{j,m+1}$ for some j, $1 \leq j \leq k(m+1)$ such that $T \cap T_{m+1} \cap A_n$ is nonempty for all n. If for each j, $1 \leq j \leq k(m+1)$, $T \cap S_{j,m+1} \cap A_n$ is empty for some n, then for n large enough, $T \cap A_n = T \cap A_{m+1} \cap A_n = \bigcup_{j=1}^{k(m+1)} T \cap S_{j,m+1} \cap A_n$ is empty, a contradiction. Thus for some j, $1 \leq j \leq k(m+1)$, $T \cap S_{j,m+1} \cap A_n$ is nonempty for all n. We define $T_{m+1} = S_{j,m+1}$, and the induction is complete.

By our inductive process we know that $\bigcap_{j=1}^{n} T_j$ is nonempty and contained in A_n. Select $x_n \in \bigcap_{j=1}^{n} T_j$. Since diam $\left(\bigcap_{j=1}^{m} T_j\right) \leq$ diam $(T_m) \leq d_m + \varepsilon_m$, we see that for j and $k \geq m$, $d(x_j, x_k) \leq d_m + \varepsilon_m$. Thus $\{x_j\}$ is a CAUCHY sequence and has a limit x. Since $x_j \in A_m$ for $j \geq m$, and since A_m is closed, $x \in A_m$. It follows that $x \in \bigcap_{m\geq 1} A_m = A_\infty$.

It remains to show that A_n approaches A_∞ in the HAUSDORFF metric. Suppose not. Then for some $r > 0$, we can find a subsequence A_{n_i} such that $B_i \equiv A_{n_i} \cap N_r(A_\infty)'$ is nonempty, where $N_r(A_\infty)'$ denotes the complement of $N_r(A_\infty)$. Since B_i is a decreasing sequence of nonempty closed sets with $\lim \gamma(B_i) = 0$, what we have already proved shows $\bigcap_{i\geq 1} B_i$ is nonempty. But this is a contradiction, since $\bigcap_{i\geq 1} B_i \subset A_\infty$ and $\bigcap_{i\geq 1} B_i \subset N_r(A_\infty)'$. Q.E.D.

If we specialize still further and assume that X is a BANACH space, we obtain results which will be crucial in our further work. The next two propositions are due to DARBO and can be found in his article [12]. The proofs are not difficult.

maps » considered in [3] and [36] (see also [38] and the remark at the end of this paper).

The reader who is only intersted in degree theory can avoid considerable effort. For degree theory the results of Section B are unnecessary; one only needs DUGUNDJI'S theorem [15]. The technical difficulties encountered in Section F in the general case completely vanish for this special case. Finally, for degree theory one can assume throughout that the metric ANR's considered are closed, convex subsets of Banach spaces.

Aside from the material on condensing maps, the theorems given here were proved in the author's Ph.D. dissertation (University of Chicago, 1969) and were summarized in [28].

A. - The measure of noncompactness and k-set-contractions.

Let us begin with some basic notions. If (A, ρ) is a bounded metric space, we define $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d > 0 \mid A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$. The idea of measure of noncompactness is due to KURATOWSKI [20]. If (X, ρ) is a metric space and A is a bounded subset of A, A inherits a metric from X, and we can speak of $\gamma(A)$. Alternatively in this case, we can define $\gamma'(A) = \inf\{d > 0 \mid A$ can be covered by a finite number of sets in X (sets not necessarily contained in A) of diameter less than or equal to $d\}$ and it is clear that $\gamma'(A) = \gamma(A)$. In practice, when we speak of measure of noncompactness, it will almost invariably be the measure of noncompactness of a bounded subset A of a metric space (X, ρ).

In this generality little can be said about the measure of noncompactness. Our first proposition lists the essentials. We leave the proof to the reader.

PROPOSITION 1. - Let (X, ρ) be a metric space and A and B bounded subsets of X. We write $N_r(A) = \{x \in X : \rho(x, A) < r\}$. Then we have a) if $A \subset B$, $\gamma(A) \leq \gamma(B)$ b) $\gamma(N_r(A)) \leq \gamma(A) + 2r$ c) if $\mathrm{cl}(A)$ denotes the closure of A, $\gamma(\mathrm{cl}(A)) = \gamma(A)$ d) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$.

If we assume that (X, ρ) is a complete metric space, we can say considerably more. Recall that in a complete metric space (X, ρ) a subset A has compact closure iff it is totally bounded, and clearly A is totally bounded iff $\gamma(A) = 0$. We also know that a decreasing sequence of compact, nonempty spaces has nonempty intersection. The following theorem, due to KURATOWSKI [20], generalizes the latter result. We include a proof for completeness.

PROPOSITION 2. (KURATOWSKI) - Let (X, ρ) be a complete metric space and let $A_1 \supset A_2 \supset \ldots$ be a decreasing sequence of nonempty, closed subsets of X.

In this paper we shall give a different generalization of the LERAY-SCHAUDER degree. We shall consider maps of the form $I - f$, where f is a « k-set-contraction », $k < 1$, and define a degree theory for them. The simplest nontrivial example of a k-set-contraction, $k < 1$, is a map of the form $U + C$, U a strict contraction defined on some subset of a Banach space, and C a compact map. Actually, we shall consider maps $f: G \to X$, where f is a k-set-contraction, $k < 1$, and G is an open subset of a « nice » metric ANR X; and we shall define a fixed point index $i_X(f, G)$. We shall prove that our definition agrees with the classical one when X is a compact polyhedron, and we shall show that all the properties of the usual fixed point index — the additivity, homotopy, aud normalization properties — extend in the appropriate way to our context. Our fixed point index will give as a special case a degree theory for maps of the form $I - f$.

Because of problems of length we cannot give applications of our fixed point index in this paper though some of the standard applications will be clear to anyone familiar with LERAY-SCHAUDER degree. In [32] we apply our fixed point index in its full generality in order to obtain asympotic fixed point theorens for k-set-contractions. These results generalize theorems of BROWDER in [8]. In [33] we specialize to the case of degree theory. We prove an invariance of domain theorem for our maps, we investigate questions of A-properness and obtain results along the lines of BROWDER and PETRYSHYN in [11], and we prove that our degree theory agrees with that defined by BROWDER and NUSSBAUM [10] where they coincide. Some of these results have been summarized in [28]. In a future paper we shall give applications to an existence theorem for a hyperbolic partial differential equation and local existence theorems for equations of evolution and functional differential equations. Preliminary results along this line are given in [30] and [31].

Some remarks about the organization of this paper: Section A gives definitions, notations, and basic theorems from the literature, though some of the results are new. In Section B we study geometrical properties of finite unions of convex sets in a vector space; our results can be viewed as a generalization of DUGUNDJI'S theorem [15] that a closed, convex subset C of a Banach space X is a retract of X. Section C presents extensions of the classical fixed point index which are necessary for the later work. In Section D the fixed point index is defined for a class of maps which includes k-set-contractions, $k < 1$. This class, though somewhat clumsy to work with, is actually quite useful in applications and provides greater flexibility than the k-set-contractions. See [30], for instance. In Section E we apply our previous work to local strict-set-contractions and spend considerable space in relating LERAY'S generalized LEFSCHETZ number [23] to our fixed point index. In the final section we use a sort of limit argument to define a fixed point index for certain 1-set-contractions and especially for the « condensing

NUSSBAUM, ROGER D.
1971
Annali di Matematica pura ed applicata
(IV), Vol. LXXXIX, pp. 217-258

The Fixed Point Index for Local Condensing Maps.

by ROGER D. NUSSBAUM (New Jersey, U.S.A.) (*)

Summary. - *We define below a fixed point index for local condensing maps f defined on open subset of «nice» metric ANR's. We prove that all the properties of classical fixed point index for continuous maps defined in compact polyhedra have appropriate generalizations. If our map is compact (a special case of a condensing map) and defined on an open subset of a Banach space, we prove that our fixed point index agrees with Leray-Schauder degree.*

Introduction.

Let X be a Banach space, G an open subset of X, and $f : \bar{G} \to X$ a compact map. For some $a \in X$ suppose that $(I-f)^{-1}(a)$ is a compact subset of G. Under these assumptions LERAY and SCHAUDER [24], using results of BROUWER [4], defined the topological degree of $I-f$ on G over the point a, $\deg(I-f, G, a)$. The LERAY-SCHAUDER degree has proved to be one of the most powerful and subtle tools for the study of fixed points of f or of the properties of the map $I-f$, f compact. It has had significant applications to partial differential equations, ordinary differential equations, and integral equations.

It was realized later that the so-called topological fixed point index (discussed in varying degrees of generality in [1], [6], [14], [22], and [37]) could be used to obtain the LERAY-SCHAUDER degree. If C is a «nice» compact, HAUSDORFF space (for instance, a compact polyhedron or a finite union of compact convex sets in a Banach space), 0 is an open subset of C, and $g : 0 \to C$ is a continuous map such that $\{x \in 0 : g(x) = x\}$ is compact or empty, then the topological fixed point index of g, $i_C(g, 0)$, is defined. This integer reduces to the Lefschetz number when $C = 0$. If f and G are as above and G is bounded, then if we set $C =$ convex closure of $\{f(x) + a : x \in G\}$, $0 = G \cap C$, and $g(x) = f(x) + a$ for $x \in 0$, $\deg(I-f, G, a) = i_C(g, 0)$.

In recent years it has become clear that it is desirable to extend degree theory to much more general maps than compact displacements of the identity. Extensions along this line have been given by BROWDER and NUSSBAUM in [10]; an exposition of this work is given by BROWDER in [9].

(*) Entrata in Redazione il 12 ottobre 1970.

ANNALI DI MATEMATICA
PURA ED APPLICATA

Organo della Fondazione «Annali di matematica pura ed applicata» D. P. R., 9 gennaio 1970, n. 44.

Questo giornale, il più antico periodico scientifico d'Italia, pubblica soltanto memorie originali, opera di collaboratori italiani o stranieri: esse vengono stampate in lingua italiana, inglese, francese o tedesca.

La pubblicazione è giunta attualmente alla quarta serie:

I Serie - pubblicata a Roma dal 1850 al 1866, constava di 15 volumi, completamente esauriti.

II Serie - pubblicata a Milano dal 1867 al 1897, consta di 26 volumi tutti esauriti. Della I e II Serie venne pubblicato nel 1904 l'Indice Generale. Sono disponibili solo alcuni fascicoli sciolti.

III Serie - pubblicata a Milano dal 1898 al 1922 consta di 31 volumi tutti esauriti, due dei quali (XX e XXI) dedicati alla memoria del matematico Lagrange, in occasione del centenario della sua morte, nel 1913. Sono disponibili alcuni fascicoli sciolti.

IV Serie - si pubblica a Bologna dal novembre 1923. Sono in vendita i Volumi dall'I all'LXXXVIII.

Per l'acquisto dei volumi e fascicoli disponibili rivolgersi alla Casa Editrice Nicola Zanichelli in Bologna (40126), Via Irnerio, 34.

Per la Redaz. e per la parte scientifica, dirigersi ad uno dei componenti il Comit. di Direz.

Per ogni fatto pertinente all'Amministrazione dirigere esclusivamente

a

NICOLA ZANICHELLI EDITORE IN BOLOGNA

Prezzo di ogni volume: L. 8.000 in Italia - L. 9.000 all'Estero

NORME PER GLI AUTORI

I manoscritti vanno diretti ad uno dei componenti il Comitato di Direzione. Si desidera che i manoscritti siano perfettamente leggibili e possibilmente scritti a macchina; per i lavori non redatti in lingua italiana, la scrittura a macchina è indispensabile. Si rivolge viva preghiera agli Autori di usare la massima discrezione nelle modificazioni, ora costosissime, sulle bozze di stampa: le correzioni non puramente tipografiche essendoci addebitate dalla Stamperia.

Per ogni lavoro sono date complessivamente N. 100 copie estratti, gratuite. Volendone un numero maggiore o con copertina speciale, rivolgersi direttamente alla Società Tipografica Editoriale Azzoguidi Via Emilia Ponente, 421-B, 40132 Bologna (Italia).

ANNALI DI MATEMATICA
PURA ED APPLICATA

Serie IV - Tomo LXXXIX - 1971
(CLXI della Raccolta)
Sotto gli auspici del Consiglio Nazionale delle Ricerche

BOLOGNA
NICOLA ZANICHELLI EDITORE
1971